The self-dual point of the two-dimensional random-cluster model is critical for $q \geqslant 1$

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Abstract

We prove a long-standing conjecture on random-cluster models, namely that the critical point for such models with parameter $q \geqslant 1$ on the square lattice is equal to the self-dual point $p_{sd}(q) = \sqrt{q}/(1+\sqrt{q})$. This gives a proof that the critical temperature of the q-state Potts model is equal to $\log(1+\sqrt{q})$ for all $q \geqslant 2$. We further prove that the transition is sharp, meaning that there is exponential decay of correlations in the sub-critical phase. The techniques of this paper are rigorous and valid for all $q \geqslant 1$, in contrast to earlier methods valid only for certain given q. The proof extends to the triangular and the hexagonal lattices as well.

Introduction

Since random-cluster models were introduced by Fortuin and Kasteleyn in 1969 [12], they have become an important tool in the study of phase transitions. The spin correlations of Potts models are rephrased as cluster connectivity properties of their random-cluster representations. This allows the use of geometric techniques, thus leading to several important applications. Nevertheless, only a few aspects of the random-cluster models are understood in full generality.

The *random-cluster model* on a finite connected graph is a model on the edges of the graph, each one being either closed or open. The probability of a configuration is proportional to

$$p^{\#\text{ open edges}}(1-p)^{\#\text{ closed edges}}q^{\#\text{ clusters}},$$

where the *edge-weight* $p \in [0,1]$ and the *cluster-weight* $q \in (0,\infty)$ are the parameters of the model. For $q \geqslant 1$, this model can be extended to infinite-volume lattices where it exhibits a phase transition at some critical parameter $p_c(q)$ (depending on the lattice). There are no general conjectures for the value of the critical point.

However, in the case of planar graphs, there is a connection (related to the Kramers-Wannier duality for the Ising model [24]) between random-cluster models on a graph and on its dual with the same cluster-weight q and appropriately related edge-weights p and $p^\star = p^\star(p)$. This relation leads in the particular case of $\mathbb{Z}^2$ (which is isomorphic to its dual) to a natural conjecture: the critical point is the same as the so-called *self-dual point* satisfying $p_{sd} = p^\star(p_{sd})$, which has a known value

$$p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

In the present article, we prove this conjecture for all $q \geqslant 1$:

Theorem 1. *Let $q \geqslant 1$. The critical point $p_c = p_c(q)$ for the random-cluster model with cluster-weight q on the square lattice $\mathbb{Z}^2$ satisfies*

$$p_c = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

A rigorous derivation of the critical point was previously known in three cases. For $q = 1$, the model is simply bond percolation, proved by Kesten in 1980 [23] to be critical at $p_c(1) = 1/2$. For $q = 2$, the self-dual value corresponds to the critical temperature of the Ising model, as first derived by Onsager in 1944 [27]; One can actually couple realizations of the Ising and FK models to relate their critical points, see [18] and references therein for details. For modern proofs in that case, see [1] or the short proof of [3]. Finally, for sufficiently large q, a proof is known based on the fact that the random-cluster model exhibits a first order phase transition (see [25, 26], the proofs are valid for q larger than 25.72). We mention that physicists derived the critical temperature for the Potts models with $q \geqslant 4$ in 1978, using non-geometric arguments based on analytic properties of the Hamiltonian [20].

In the sub-critical phase, we prove that the probability for two points x and y to be connected by a path decays exponentially fast with respect to the distance between x and y. In the super-critical phase, the same behavior holds in the dual model. This phenomenon is known as a *sharp phase transition*:

Theorem 2. *Let $q \geqslant 1$. For any $p < p_c(q)$, there exist $0 < C(p, q), c(p, q) < \infty$ such that for any $x, y \in \mathbb{Z}^2$,*

$$\phi_{p,q}(x \leftrightarrow y) \leqslant C(p, q)\mathrm{e}^{-c(p,q)|x-y|}, \tag{0.1}$$

where $|\cdot|$ denotes the Euclidean norm.

The proof involves two main ingredients. The first one is an estimate on crossing probabilities at the self-dual point $p = \sqrt{q}/(1 + \sqrt{q})$: the probability of crossing a rectangle with aspect ratio $(\alpha, 1)$ — meaning that the ratio between the width and the height is of order α — in the horizontal direction is bounded away from 0 and 1 uniformly in the size of the box. This result is the main new contribution of this paper. It is a generalization of the celebrated Russo-Seymour-Welsh theorem for percolation.

The second ingredient is a collection of sharp threshold theorems, which were originally introduced for product measures. They have been used in many contexts, and are a powerful tool for the study of phase transitions, see Bollobás and Riordan [5, 6]. These theorems were later extended to positively associated measures by Graham and Grimmett [15, 16, 18]. In our case, they may be used to show that the probability of crossings goes to 1 when $p > \sqrt{q}/(1+\sqrt{q})$.

Actually, the situation is slightly more complicated than usual: the dependence inherent in the model makes boundary conditions difficult to handle, so that new arguments are needed. More precisely, one can use a classic sharp threshold argument for symmetric increasing events in order to deduce that the crossing probabilities of larger and larger domains, under *wired boundary condition*, converge to 1 whenever $p > \sqrt{q}/(1 + \sqrt{q})$. Moreover, the theorem provides us with bounds on the speed of convergence for rectangles with *wired boundary condition*. A new way of combining long paths allows us to create an infinite cluster. We emphasize that the classic construction, used by Kesten [23] for instance, does not seem to work in our case.

The approach allows the determination of the critical value, but it provides us with a rather weak estimate on the speed of convergence for crossing probabilities. Nevertheless, combining the fact that the crossing probabilities go to 0 when $p < p_{sd}$ with a very general threshold theorem, we deduce that the cluster-size at the origin has finite moments of any order. It is then an easy step to derive the exponential decay of the two-point function.

Theorem 1 has several notable consequences. First, it extends up to the critical point results that are known for the sub-critical random-cluster models under the exponential decay condition (for instance, Ornstein-Zernike estimates [8] or strong mixing properties). Second, it identifies the critical value of the Potts models via the classical coupling between random-cluster models with cluster-weight $q \in \mathbb{N}$ and the q-state Potts models:

Theorem 3. *Let $q \geqslant 2$ be an integer; consider the q-state Potts model on $\mathbb{Z}^2$, defined by the Hamiltonian*

$$H_q(c) := -\sum_{(xy)\in E} \delta_{c_x,c_y}$$

(where $c_x \in \{1,\dots,q\}$ is the color at site x and E the set of edges of the lattice). The model exhibits a phase transition at the critical inverse temperature

$$\beta_c(q) = \log(1+\sqrt{q}).$$

The methods of this paper harness symmetries of the graph, together with the self-dual property of the square lattice. In the case of the hexagonal and triangular lattices, the symmetries of the graphs, the duality property between the hexagonal and the triangular lattices and the star-triangle relation allow us to extend the crossing estimate proved in Section 2, at the price of additional technical difficulties. The rest of the proof can be carried over to the triangular and the hexagonal lattices as well, yielding the following result:

Theorem 4. *The critical value $p_c = p_c(q)$ for the random-cluster model with cluster-weight $q \geqslant 1$ satisfies*

$$\begin{array}{ll} y_c^3 + 3y_c^2 - q = 0 & \textit{on the triangular lattice and} \\ y_c^3 - 3qy_c - q^2 = 0 & \textit{on the hexagonal lattice,} \end{array}$$

where $y_c := p_c/(1-p_c)$. Moreover, there is exponential decay in the sub-critical phase.

There are many unanswered questions related to Theorem 1. First, the behavior at criticality is not well understood in the general case, and it seems that new techniques are needed. It is conjectured that the random-cluster model undergoes a second-order phase transition for $q \in (0,4)$ (it is further believed that the scaling limit is then conformally invariant), and a first-order phase transition when $q \in (4,\infty)$. Proving the above for every q remains a major open problem. We mention that the random-cluster models with parameter $q = 2$ [29] or q very large [25, 26] are much better understood. Second, the technology developed in the present article relies heavily on the positive association property of the random-cluster measures with $q \geqslant 1$. Our strategy does not extend to random-cluster models with $q < 1$. Understanding these models is a challenging open question.

The paper is organized as follows. In Section 1 we review some basic features of random-cluster models. Section 2 is devoted to the statement and the proof of the crossing estimates. In Section 3, we briefly present the theory of sharp threshold that we will employ in the next section. Section 4 contains the proofs of Theorems 1 and 2. Section 5 is devoted to extensions to other lattices and contains the proof of Theorem 4.

1 Basic features of the model

We start with an introduction to the basic features of random-cluster models; more detail and proofs can be found in Grimmett's monograph [18].

Definition of the random-cluster model. The random-cluster measure can be defined on any graph. However, we will restrict ourselves to the square lattice (of mesh size 1), or more precisely a version rotated by an angle $\pi/4$, see Figure 1. We denote this lattice by $\mathbb{L} = (\mathbb{V},\mathbb{E})$, with $\mathbb{V}$ denoting the set of *sites* and $\mathbb{E}$ the set of *edges*. In this paper, G will always denote a connected subgraph of $\mathbb{L}$, *i.e.* a subset of vertices of $\mathbb{V}$ together with all the edges between them. We denote by ∂G the boundary of G, *i.e.* the set of sites of G linked by an edge of $\mathbb{E}$ to a site of $\mathbb{V} \setminus G$.

A *configuration* ω on G is a subgraph of G, composed of the same sites and a subset of its edges. We will call the edges belonging to ω *open*, the others *closed*. Two sites a and b are said to be *connected* if there is an *open path*, *i.e.* a path composed of open edges only, connecting them (this event will be denoted by $a \leftrightarrow b$). Two sets A and B are *connected* if there exists an open path connecting them (denoted $A \leftrightarrow B$). The maximal connected components will be called *clusters*. We will often simply use the term *path* for *open path* when there is no possible ambiguity.

A *boundary condition* ξ is a partition of ∂G. We denote by $\omega \cup \xi$ the graph obtained from the configuration ω by identifying (or *wiring*) the edges in ξ that belong to the same component of ξ. Boundary conditions should be understood as encoding how sites are connected outside G. Let $o(\omega)$ (resp. $c(\omega)$) denote the number of open (resp. closed) edges of ω and $k(\omega, \xi)$ the number of connected components of $\omega \cup \xi$. The probability measure $\phi^{\xi}_{p,q,G}$ of the random-cluster model on G with parameters p and q and boundary condition ξ is defined by

$$\phi^{\xi}_{p,q,G}(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}}{Z^{\xi}_{p,q,G}} \tag{1.1}$$

for every configuration ω on G, where $Z^{\xi}_{p,q,G}$ is a normalizing constant referred to as the *partition function*.

Let $t < x$ and $y < z$; we will identify the rectangle $[t, x) \times [y, z)$ with the set of vertices in $\mathbb{V}$ that lie within it. The graph with vertex set $[t, x) \times [y, z)$ together with the induced subset of $\mathbb{E}$ is called a *rectangle of* $\mathbb{L}$.

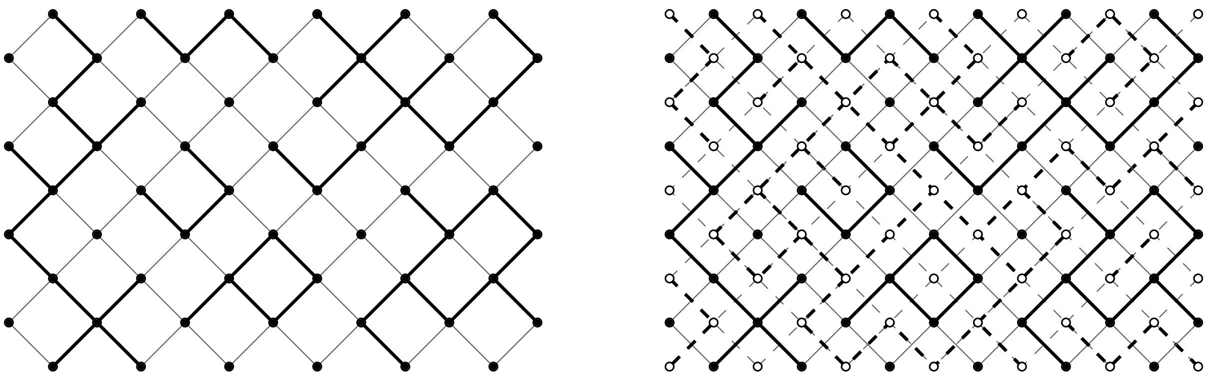

Figure 1: Left: Example of a configuration on the rotated lattice. **Right:** A configuration together with its dual configuration.

The finite energy property. This is a very simple property of random-cluster models. Let $\varepsilon \in (0, 1/2)$. The conditional probability for an edge to be open, knowing the states of all the other edges, is bounded away from 0 and 1 uniformly in $p \in (\varepsilon, 1-\varepsilon)$ and in the configuration away from this edge. This property extends to any finite family of edges.

The domain Markov property. One can encode, using appropriate boundary condition ξ, the influence of the configuration outside a sub-graph on the measure within it. Consider a graph $G = (V, E)$ and a random-cluster measure $\phi^{\psi}_{p,q,G}$ on it. For $F \subset E$, consider G' with F as the set of edges and the endpoints of it as the set of sites. Then, the restriction to G' of $\phi^{\psi}_{p,q,G}$ conditioned to match some configuration ω outside G' is exactly $\phi^{\xi}_{p,q,G'}$, where ξ describes the connections inherited from $\omega \cup \psi$ (two sites are wired if they are connected by a path in $\omega \cup \psi$ outside the box).

The FKG inequality and comparison between boundary conditions. An event is called *increasing* if it is preserved by addition of open edges, see [18]. Random-cluster models with parameter $q \geqslant 1$ are *positively correlated.* This property has two important consequences, the first one being the *Fortuin-Kasteleyn-Ginibre inequality*:

$$\phi^{\xi}_{p,q,G}(A \cap B) \geqslant \phi^{\xi}_{p,q,G}(A)\phi^{\xi}_{p,q,G}(B), \tag{1.2}$$

which holds for every pair of increasing events A and B and any boundary conditions ξ. This correlation inequality is extremely important in the study of random-cluster models.

The second property is a *comparison between boundary conditions*: for any boundary conditions $\psi \leqslant \xi$, meaning that sites wired in ψ are wired in ξ, we have

$$\phi^{\psi}_{p,q,G}(A) \leqslant \phi^{\xi}_{p,q,G}(A) \tag{1.3}$$

for any increasing event A. We say that $\phi^{\xi}_{p,q,G}$ *stochastically dominates* $\phi^{\psi}_{p,q,G}$. Combined with the domain Markov property, the comparison between boundary conditions allows to give bounds on conditional probabilities.

Examples of boundary conditions: free, wired and periodic. Two boundary conditions play a special role in the study of the random-cluster model. The *wired* boundary condition, denoted by $\phi^{1}_{p,q,G}$, is specified by the fact that all the vertices on the boundary are pairwise wired (only one set in the partition). The *free* boundary condition, denoted by $\phi^{0}_{p,q,G}$, is specified by no wiring between sites. These boundary conditions are extremal for the stochastic ordering, since any boundary condition has fewer (resp. more) wired sites than in the wired (resp. free) boundary condition.

We will also consider *periodic* boundary conditions: for $n \geqslant 1$ (not necessarily integer), the torus of size n can be seen as the box $[0,n)^2$ with the boundary condition obtained by imposing that $(i,0)$ is wired to (i,n) for every $i \in [0,n]$ and that $(0,j)$ is connected to (n,j) for every $j \in [0,n]$. We will denote the random-cluster measure on the torus of size n by $\phi^{\mathrm{p}}_{p,q,[0,n]^2}$ or more concisely $\phi^{\mathrm{p}}_{p,q,n}$. Note that this realization of the torus provides us with a natural embedding in the plane (though of course the boundary condition cannot be realized using disjoint paths outside the square $[0,n]^2$ because the torus itself is not a planar graph).

Dual graph and planar duality. In two dimensions, one can associate with any random-cluster model a dual model. Let G be a finite graph embedded in a surface. Define the dual graph $G^\star = (V^\star, E^\star)$ in the usual way as follows: place a dual site at the centers of the faces of G (the external face, when considering a graph on the plane, must be counted as a face of the graph), and for every bond $e \in E$, place a dual bond between the two dual sites corresponding to faces bordering e. Given a subgraph configuration ω, construct a bond model on $G^\star$ by declaring any bond of the dual graph to be open (resp. closed) if the corresponding bond of the primal lattice is closed (resp. open) for the initial configuration. The new configuration is called the *dual configuration* of ω.

When defining the dual of the FK model, one must be careful about boundary conditions (it will be crucial in this article). We first recall the classic case: consider the random-cluster measure with parameters (p,q) on the square of size n, with wired boundary conditions — which can be realized as an FK model on a slightly larger square, conditioned to have all the bonds outside the smaller square open. The dual model on the dual graph (which is a square with an additional outer vertex) given by the dual configurations then corresponds to a random-cluster measure with free boundary conditions, with the same parameter q and a dual parameter $p^\star = p^\star(p,q)$ satisfying

$$p^\star(p,q) := \frac{(1-p)q}{(1-p)q+p}, \text{ or equivalently } \frac{p^\star p}{(1-p^\star)(1-p)} = q.$$

In other words, the dual measure $(\phi^1_{p,q,n})^\star$ of $\phi^1_{p,q,n}$ is $\phi^0_{p^\star,q,n-1}$. This relation is an instance of *planar duality*. It is then natural to define the self-dual point $p_{sd} = p_{sd}(q)$ by solving the equation $p^\star(p_{sd}, q) = p_{sd}$, thus obtaining

$$p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{1.4}$$

Similarly, the dual of a random-cluster model with parameters (p, q) and free boundary conditions is a random-cluster model with parameters $(p^\star, q)$ and wired boundary conditions.

Planar duality for periodic boundary conditions. The case of periodic boundary conditions, or equivalently the case of the random-cluster model defined on a torus (with no boundary condition) is a little more involved: indeed, its dual is *not* a random-cluster model; but it is not very different from one, and that will be enough for our purposes. To state duality in this case, we need additional notations. Recall that if ω is a configuration, $o(\omega)$ stands for the number of open bonds in ω, $c(\omega)$ for the number of closed bonds and $k(\omega)$ for the number of connected components of ω; let $f(\omega)$ be the number of faces delimited by ω, *i.e.* the number of connected components of the complement of the set of open bonds, and $s(\omega)$ be the number of vertices in the underlying graph (it does not depend on ω). We will now define an additional parameter $\delta(\omega)$.

Call a (maximal) connected component of ω a *net* if it contains two non-contractible simple loops of different homotopy classes, and a *cycle* if it is non-contractible but is not a net. Notice that every configuration ω can be of one of three types:

- One of the clusters of ω is a net. Then no other cluster of ω can be a net or a cycle. In that case, we let $\delta(\omega) = 2$;
- One of the clusters of ω is a cycle. Then no other cluster can be a net, but other clusters can be cycles as well (in which case all the involved, simple loops are in the same homotopy class). We then let $\delta(\omega) = 1$;
- None of the clusters of ω is a net or a cycle. We let $\delta(\omega) = 0$.

With this additional notation, Euler's formula becomes

$$s(\omega) - o(\omega) + f(\omega) = k(\omega) + 1 - \delta(\omega). \tag{1.5}$$

Besides, these terms transform in a simple way under duality: $o(\omega) + o(\omega^\star)$ is a constant, $f(\omega) = k(\omega^\star)$ and $\delta(\omega) = 2 - \delta(\omega^\star)$. The same proof as that of usual duality, taking the additional topology into account, then leads to the relation

$$(\phi^{\mathrm{p}}_{p,q,n})^\star(\{\omega\}) \propto q^{1-\delta(\omega)} \phi^{\mathrm{p}}_{p^\star,q,n}(\{\omega\}). \tag{1.6}$$

This means that even though the dual model of the periodic boundary condition FK model is not exactly an FK model at the dual parameter, it is absolutely continuous with respect to it and the Radon-Nikodym derivative is bounded above and below by constants depending only on q. Another way of stating the same result would be to define a *balanced FK model* with weights

$$\tilde{\phi}^{\mathrm{p}}_{p,q,n}(\{\omega\}) = \frac{(\sqrt{q})^{1-\delta(\omega)}}{Z} \phi^{\mathrm{p}}_{p,q,n}(\{\omega\}) :$$

this one is absolutely continuous with respect to the usual FK model and does satisfy exact duality.

Infinite-volume measures and the definition of the critical point. The domain Markov property and comparison between boundary conditions allow us to define infinite-volume measures. Indeed, one can consider a sequence of measures on boxes of increasing size with free boundary conditions. This sequence is increasing in the sense of stochastic domination, which implies that it converges weakly to a limiting measure, called the random-cluster measure on $\mathbb{L}$ with free boundary condition (and denoted by $\phi^0_{p,q}$). This classic construction can be performed with many other sequences of measures, defining several *a priori* different infinite-volume measures on $\mathbb{L}$. For instance, one can define the random-cluster measure $\phi^1_{p,q}$ with wired boundary condition, by considering the decreasing sequence of random-cluster measures on finite boxes with wired boundary condition.

For given $q \geqslant 1$, it is known that uniqueness can fail only for p in a countable set $\mathcal{D}_q$, see Theorem (4.60) of [18]. Therefore, there exists a *critical point* p_c such that for any infinite-volume measure with $p < p_c$ (resp. $p > p_c$), there is almost surely no infinite component of connected sites (resp. at least one infinite component).

Remark. It is natural to conjecture that the critical point satisfies $p_c = p_{sd}$. Indeed, if one assumes $p_c \neq p_{sd}$, there would be two phase transitions, one at p_c, due to the change of behavior in the primal model, and one at $p_c^\star$, due to the change of behavior in the dual model.

The inequality $p_c \geqslant p_{sd}$. As in the case of percolation, a lower bound for the critical value can be derived using the uniqueness of the infinite cluster above the critical point. Indeed, if one assumes that $p_c < p_{sd}$, the configuration at p_{sd} must contain one infinite open cluster and one infinite dual open cluster (since the random-cluster model in the dual is then super-critical as well). Intuition indicates that such coexistence would imply that there is more than one infinite open cluster; an elegant argument (due to Zhang in the case of percolation) formalizes this idea. We refer to the exposition in Theorem (6.17) of [18] for full detail, but still give a sketch of the argument.

The proof goes as follows, see Figure 2. Assume that $p_c < p_{sd}$ and consider the random-cluster model with $p = p_{sd}$. There is an infinite open cluster, and therefore, we can choose a large box such that the infinite open cluster and the dual infinite open cluster touch the boundary with probability greater than $1-\varepsilon$. The FKG inequality (through the so-called "square-root trick": for two increasing events A and B having the same probability,

$$\phi^\xi_{p,q,G}(A) \geqslant 1 - \Big(1 - \phi^\xi_{p,q,G}(A \cup B)\Big)^{1/2}$$

with similar formulas when more events are involved) implies that the infinite open cluster actually touches the top side of the box, using only edges outside the box, with probability greater than $1-\varepsilon^{1/4}$. We deduce that with probability at least $1-2\varepsilon^{1/4}$, the infinite open cluster touches both the top and bottom sides, using only edges outside of the box.

A similar argument implies that the infinite dual open cluster touches both the left- and right-hand sides of the box with probability at least $1-2\varepsilon^{1/4}$. Therefore, with probability at least $1-4\varepsilon^{1/4}$, the complement of the box contains an infinite open path touching the top of the box, one touching the bottom, and infinite dual open paths touching each of the vertical edges. Enforcing edges in the box to be closed, which brings only a positive multiplicative factor due to the finite energy property of the model, and choosing ε sufficiently small, we deduce that there are two infinite open clusters with positive probability. Since the infinite open cluster must be unique (see [18] again), this is a contradiction, which implies that $p_c \geqslant p_{sd}$.

When $p < p_{sd} \leqslant p_c$, there is no infinite cluster for any infinite-volume measure. General arguments imply uniqueness of the infinite-volume measure whenever $p \neq p_{sd}$ and $q \geqslant 1$ (see Theorem (6.17) of [18]). This fact will be useful in the sequel since, *except at criticality*, we do not have to specify which infinite-volume measure is under consideration. We will denote the unique infinite-volume measure by $\phi_{p,q}$ when $p \neq p_{sd}$.

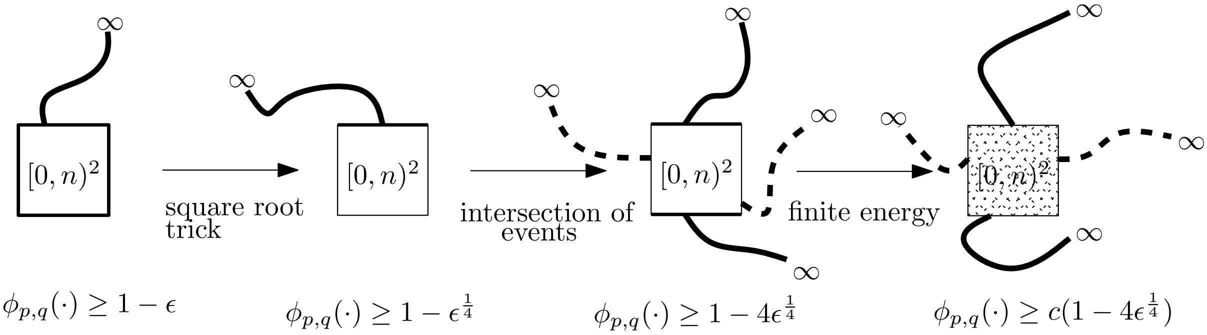

Figure 2: The steps of the proof that $p_c \geqslant p_{sd}$.

2 Crossing probabilities for rectangles at the self-dual point

In this section, we prove crossing estimates for rectangles of prescribed aspect ratio. This is an extension of the Russo-Seymour-Welsh theory for percolation. We will work with $p = p_{sd}(q)$ and the measures $\phi^1_{p_{sd},q}$ and $\phi^{\mathrm{p}}_{p_{sd},q,n}$; we present the proof in the periodic case. The case of the (bulk) wired boundary condition can be derived from this case (see Corollary 9).

For a rectangle R, let $\mathcal{C}_v(R)$ denote the event that there exists a path between the top and the bottom sides which stays inside the rectangle. Such a path is called a *vertical (open) crossing* of the rectangle. Similarly, we define $\mathcal{C}_h$ to be the event that there exists an *horizontal open crossing* between the left and the right sides. Finally, $\mathcal{C}^\star_v(R^\star)$ denotes the event that there exists a dual-open crossing from top to bottom in the dual graph $R^\star$ of R.

The following theorem states that, at the self-dual point, the probability of crossing a rectangle horizontally is bounded away from 0 uniformly in the sizes of both the rectangle and the torus provided that the aspect ratio of the rectangles remains constant. The size of the ambient torus is denoted by m. Note that $p = p^\star$ when $p = p_{sd}$, and hence the balanced FK measure on the torus is self-dual.

Theorem 5. *Let $\alpha > 1$ and $q \geqslant 1$. There exists $c(\alpha) > 0$ such that for every $m > \alpha n > 0$,*

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,\alpha n) \times [0,n))\big) \geqslant c(\alpha). \tag{2.1}$$

We begin the proof with a lemma, which corresponds to the existence of $c(1)$ and is based on the self-duality of random-cluster measures on the torus. This lemma is classic and is the natural starting point for any attempt to prove RSW-like estimates.

Lemma 6. *Let $q \geqslant 1$, there exists $c(1) > 0$ (depending only on the parameter q) such that for every $m > n \geqslant 1$, $\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,n)^2)\big) \geqslant c(1)$.*

Proof. Note that the dual of $[0,n)^2$ is $[0,n)^2$ (meaning the sites of the dual torus inside $[0,n)^2$), see Figure 3. If there is no open crossing from left to right in $[0,n)^2$, there exists necessarily a dual-open crossing from top to bottom in the dual configuration. Hence, the complement of $\mathcal{C}_h([0,n)^2)$ is $\mathcal{C}^\star_v([0,n)^2)$, thus yielding

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,n)^2)\big) + \phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathcal{C}^\star_v([0,n)^2)\big) = 1.$$

Using the duality property for periodic boundary conditions and the symmetry of the lattice, the probability $\phi^{\mathrm{p}}_{p_{sd},q,m}(\mathcal{C}^\star_v([0,n)^2))$ is larger than $c\phi^{\mathrm{p}}_{p_{sd},q,m}(\mathcal{C}_h([0,n)^2))$ (for some constant c only depending on q), giving

$$1 \leqslant (1+c)\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathcal{C}_h([0,n)^2)\big),$$

which concludes the proof. □

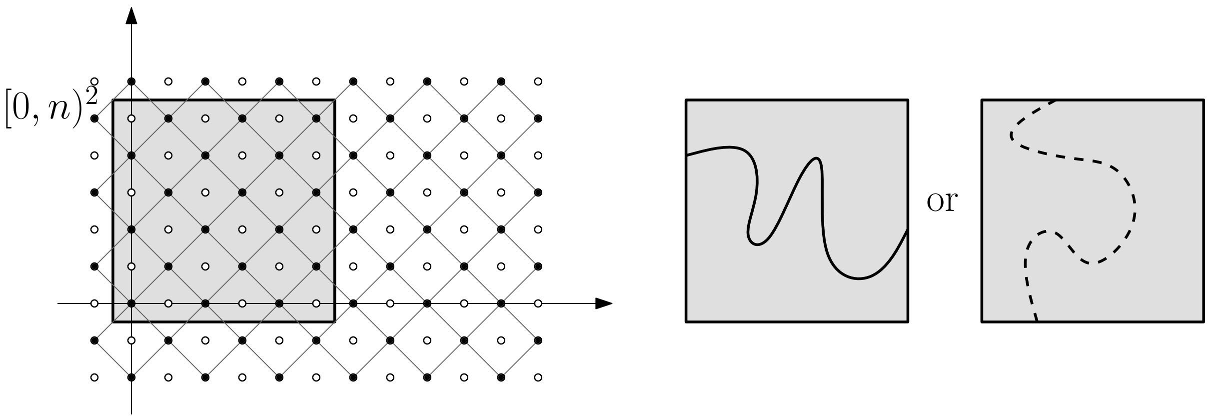

Figure 3: Left: The square $[0,n)^2$ (all the sites in the shaded region) and its dual have the same graph structure. **Right:** The events $\mathcal{C}_h([0,n)^2)$ and $\mathcal{C}_v^\star([0,n)^2)$.

Remark. This lemma could be stated in terms of the balanced FK measure instead of the usual one; then, as in the case of percolation, one would obtain that the probability of a horizontal crossing of the square is exactly 1/2. However, because going back and forth between the balanced and standard measure would be a little tedious in what follows, we choose to state it in terms of $\phi^{\mathrm{p}}_{p_{sd},q,m}$ — and to have $c(1)$ depend on the value of q.

The only major difficulty is to prove that rectangles of aspect ratio α are crossed in the horizontal direction — with probability uniformly bounded away from 0 — for *some* $\alpha > 1$. There are many ways to prove this in the case of percolation. Nevertheless, they always involve independence in a crucial way; in our case, independence fails, so we need a new argument. The main idea is to invoke self-duality in order to enforce the existence of crossings, even in the case where boundary conditions could look disadvantageous. In order to do that, we introduce the following family of domains, which are in some sense nice symmetric domains.

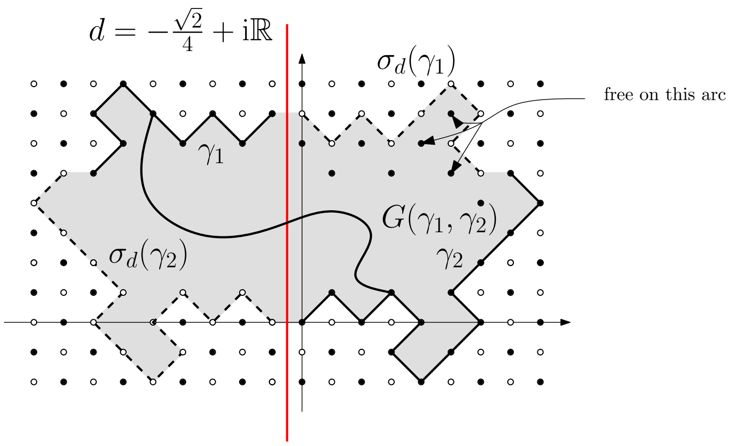

Figure 4: Two paths γ_1 and γ_2 satisfying Hypothesis $(\star)$ and the graph $G(\gamma_1,\gamma_2)$.

Define the line $d := -\sqrt{2}/4 + \mathrm{i}\mathbb{R}$. The orthogonal symmetry σ_d with respect to this line maps $\mathbb{L}$ to $\mathbb{L}^\star$. Let γ_1 and γ_2 be two paths satisfying the following Hypothesis $(\star)$, see Figure 4:

- γ_1 remains on the left of d and γ_2 remains on the right;
- γ_2 begins at 0 and γ_1 begins on a site of $\mathbb{L} \cap (-\sqrt{2}/2 + \mathrm{i}\mathbb{R}_+)$;
- γ_1 and $\sigma_d(\gamma_2)$ do not intersect (as curves in the plane);

- γ_1 and $\sigma_d(\gamma_2)$ end at two sites (one primal and one dual) which are at distance $\sqrt{2}/2$ from each other.

The definition extends trivially via translation, so we will say that the pair (γ_1, γ_2) satisfies Hypothesis $(\star)$ if one of its translations does.

When following the paths in counter-clockwise order, we can create a circuit by linking the end points of γ_1 and $\sigma_d(\gamma_2)$ by a straight line, the start points of $\sigma_d(\gamma_2)$ and γ_2, the end points of γ_2 and $\sigma_d(\gamma_1)$, and the start points of $\sigma_d(\gamma_1)$ and γ_1. The circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$ surrounds a set of vertices of $\mathbb{L}$. Define the graph $G(\gamma_1, \gamma_2)$ composed of sites of $\mathbb{L}$ that are surrounded by the circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$, and of edges of $\mathbb{L}$ that remain entirely within the circuit (boundary included).

The *mixed boundary condition* on this graph is wired on γ_1 (all the edges are pairwise connected), wired on γ_2, and free elsewhere. We denote the measure on $G(\gamma_1, \gamma_2)$ with parameters (p_{sd}, q) and mixed boundary condition by $\phi_{p_{sd},q,\gamma_1,\gamma_2}$ or more simply ϕ_{γ_1,γ_2}.

Lemma 7. *For any pair (γ_1, γ_2) satisfying Hypothesis $(\star)$, the following estimate holds:*

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geqslant \frac{1}{1+q^2}.$$

Proof. On the one hand, if γ_1 and γ_2 are not connected, $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ must be connected by a dual path in the dual model (event corresponding to $\sigma_d(\gamma_1) \leftrightarrow \sigma_d(\gamma_2)$ in the dual model). Hence,

$$1 = \phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + \sigma_d * \phi^{\star}_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2), \tag{2.2}$$

where $\sigma_d * (\phi^{\star}_{\gamma_1,\gamma_2})$ denotes the image under σ_d of the dual measure of ϕ_{γ_1,γ_2}. This measure lies on $G(\gamma_1, \gamma_2)$ as well and has parameters (p_{sd}, q).

When looking at the dual measure of a random-cluster model, the boundary condition is transposed into a new boundary condition for the dual measure. In the case of the periodic boundary condition, we obtained the same boundary condition for the dual measure. Here, the boundary condition becomes wired on $\gamma_1 \cup \gamma_2$ and free elsewhere (this is easy to check using Euler's formula).

It is very important to notice that the boundary condition is *not* exactly the mixed one, since γ_1 and γ_2 are wired *together*. Nevertheless, the Radon-Nikodym derivative of $\sigma_d * \phi^{\star}_{\gamma_1,\gamma_2}$ with respect to ϕ_{γ_1,γ_2} is easy to bound. Indeed, for any configuration ω, the number of cluster can differ only by 1 when counted in $\sigma_d * \phi^{\star}_{\gamma_1,\gamma_2}$ or ϕ_{γ_1,γ_2} so that the ratio of partition functions belongs to $[1/q, q]$. Therefore, the ratio of probabilities of the configuration ω remains between $1/q^2$ and q^2. This estimate extends to events by summing over all configurations. Therefore,

$$\sigma_d * \phi^{\star}_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \leqslant q^2 \phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2).$$

When plugging this inequality into (2.2), we obtain

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + q^2 \phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geqslant 1$$

which implies the claim. □

We are now in a position to prove the key result of this section.

Proposition 8. *For all $m > 3n/2 > 0$, the following holds:*

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big[\mathcal{C}^v\big([0,n) \times [0, 3/2n)\big)\big] \geqslant \frac{c(1)^3}{2(1+q^2)}.$$

Before proving this proposition, we show how it implies Theorem 5. The strategy is straightforward and classic: we combine crossings together, using only the FKG inequality.

Proof of Theorem 5. If $\alpha < 3/2$, Proposition 8 implies the claim so we can assume $\alpha > 3/2$. Define the following rectangles, see Figure 5:

$$R_j^h = [jn/2, jn/2 + 3n/2) \times [0, n) \quad \text{and} \quad R_j^v = [jn/2, jn/2 + n) \times [0, n)$$

for $j \in [0, \lfloor 2\alpha \rfloor - 1]$, where $\lfloor x \rfloor$ denotes the integer part of x. If every rectangle R_j^h is crossed horizontally, and every rectangle R_j^v is crossed vertically, then $[0, \alpha n) \times [0, n)$ is crossed horizontally. We denote this event by B. The rectangle R_j^h is crossed horizontally with probability greater than $c(1)^3/[2(1 + q^2)]$ (Proposition 8), the rectangle R_j^v is crossed vertically with probability greater than $c(1)$ (Lemma 6) and so, using the FKG inequality,

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathcal{C}_h([0, \alpha n) \times [0, n))\big) \geqslant \phi^{\mathrm{p}}_{p_{sd},q,m}(B) \geqslant \left(\frac{c(1)^4}{2(1 + q^2)}\right)^{\lfloor 2\alpha \rfloor}.$$

The claim follows with $c(\alpha) := [c(1)^4/(2 + 2q^2)]^{\lfloor 2\alpha \rfloor}$. □

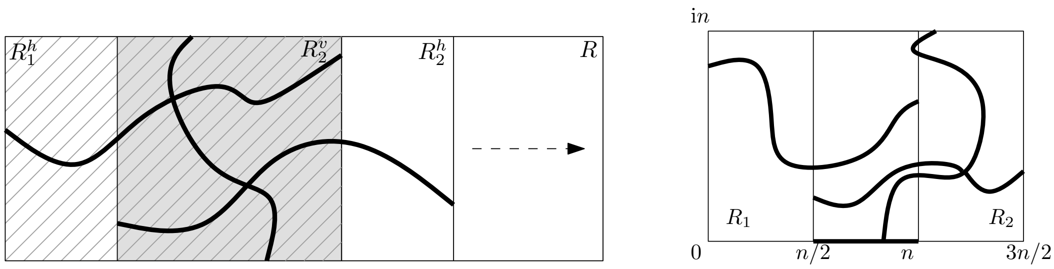

Figure 5: Left: A combination of crossings in smaller rectangles creating a horizontal crossing of a very long rectangle. **Right:** The rectangles R, R_1 and R_2 and the event A.

Proof of Proposition 8. The proof goes as follows: we start by creating two paths crossing square boxes, and we then prove that they are connected with good probability.

Setting of the proof. Consider the rectangle $R = [0, 3n/2) \times [0, n)$ which is the union of the rectangles $R_1 = [0, n) \times [0, n)$ and $R_2 = [n/2, 3n/2) \times [0, n)$, see Figure 5. Let A be the event defined by the following conditions:

- R_1 and R_2 are both crossed horizontally (these events have probability at least $c(1)$ to occur, using Lemma 6);
- $[n/2, n) \times \{0\}$ is connected inside R_2 to the top side of R_2 (which has probability greater than $c(1)/2$ to occur using symmetry and Lemma 6).

Employing the FKG inequality, we deduce that:

$$\phi^{\mathrm{p}}_{p_{sd},q,m}(A) \geqslant \frac{c(1)^3}{2}. \tag{2.3}$$

When A occurs, define Γ_1 to be the top-most horizontal crossing of R_1, and Γ_2 the right-most vertical crossing of R_2 from $[n/2, n) \times \{0\}$ to the top side. Note that this path is automatically connected to the right-hand side of R_2 — which is the same as the right-most side of R. If Γ_1 and Γ_2 are connected, then there exists a horizontal crossing of R. In the following, we show that Γ_1 and Γ_2 are connected with good probability.

Exploration of the paths Γ_1 and Γ_2. There is a standard way of exploring R in order to discover Γ_1 and Γ_2. Start an exploration from the top-left corner of R that leaves open edges on its right, closed edges on its left and remains in R_1. If A occurs, this exploration will touch the right-hand side of R_1 before its bottom side; stop it the first time it does. Note that the exploration process "slides" between open edges of the primal lattice and dual open edges of the dual (formally, this exploration process is defined on the medial lattice, see *e.g.* [3]). The open edges that are adjacent to the exploration form the top-most horizontal crossing of R_1, *i.e.* Γ_1. At the end of the exploration, the process has *a priori* discovered a set of edges which lies above Γ_1, so that the remaining part of R_1 is undiscovered.

By starting an exploration at point $(n, 0)$, leaving open edges on its left and closed edges on its right, we can explore the rectangle R_2. If A holds, the exploration ends on the top side of R_2. The open edges adjacent to the exploration constitute the path Γ_2 and the set of edges already discovered lies "to the right" of Γ_2.

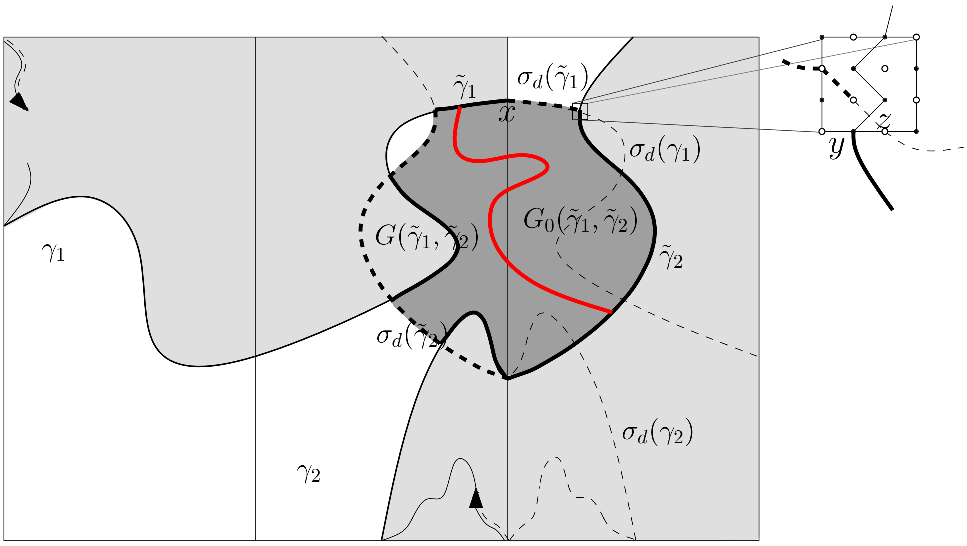

Figure 6: The light gray area is the part of R that is *a priori* discovered by the exploration processes (note that this area can be much smaller). The dark gray is the domain $G_0(\tilde\gamma_1, \tilde\gamma_2)$. We have depicted all the paths involved in the construction. Note that dashed curves are "virtual paths" of the dual lattice obtained by the reflection σ_d: they are not necessarily dual open.

The reflection argument. Assume first that we know $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ and that they do not intersect. Let x be the end-point of γ_1, *i.e.* its unique point on the right-hand side of R_1. We want to define a set $G_0(\gamma_1, \gamma_2)$ similar to those considered in Lemma 7. Apply the following "surgical procedure," see Figure 6:

- First, define the symmetric paths $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ of γ_1 and γ_2 with respect to the line $d := (n - \sqrt{2}/4) + \mathrm{i}\mathbb{R}$;
- Then, parametrize the path $\sigma_d(\gamma_1)$ by the distance (along the path) to its starting point $\sigma_d(x)$ and define $\tilde\gamma_1 \subset \gamma_1$ so that $\sigma_d(\tilde\gamma_1)$ is the part of $\sigma_d(\gamma_1)$ between the start of the path and the first time it intersects γ_2. As before, the paths are considered as curves of the plane; we denote z the intersection point of the two curves. Note that γ_1 and γ_2 are not intersecting, which forces $\sigma_d(\gamma_1)$ and γ_2 to be;
- From this, parametrize the path γ_2 by the distance to its starting point $(n, 0)$ and set y to be the last visited site in $\mathbb{L}$ before the intersection z. Define $\tilde\gamma_2$ to be the part of γ_2 between the last point intersecting $n + \mathrm{i}\mathbb{R}$ before y and y itself;

- Paths $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ satisfy Hypothesis ($\star$) so that the graph $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$ can be defined;
- Construct a sub-graph $G_0(\gamma_1, \gamma_2)$ of $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$ as follows: the edges are given by the edges of $\mathbb{L}$ included in the connected component of $G(\tilde{\gamma}_1, \tilde{\gamma}_2) \setminus (\gamma_1 \cup \gamma_2)$ (*i.e.* $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$ minus the set $\gamma_1 \cup \gamma_2$) containing d (it is the connected component which contains $x - \varepsilon\mathrm{i}$, where ε is a very small number), and the sites are given by their endpoints.

The graph $G_0(\gamma_1, \gamma_2)$ has a very useful property: none of its edges has been discovered by the previous exploration paths. Indeed, $\sigma_d(\gamma_1)$ and $\sigma_d(x)$ are included in the unexplored connected component of $R \setminus R_1$, and so does $G_0(\gamma_1, \gamma_2) \cap (R \setminus R_1)$. Edges of $G_0(\gamma_1, \gamma_2)$ in R_1 are in the same connected component of $R \setminus (\gamma_1 \cup \gamma_2)$ as $x - \varepsilon\mathrm{i}$, and thus lie 'below' γ_1.

Conditional probability estimate. Still assuming that γ_1 and γ_2 do not intersect, we would like to estimate the probability of γ_1 and γ_2 being connected by a path knowing that $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$. Following the exploration procedure described above, we can discover γ_1 and γ_2 without touching any edge in the interior of $G_0(\gamma_1, \gamma_2)$. Therefore, the process in the domain is a random-cluster model with specific boundary condition.

The boundary of $G_0(\gamma_1, \gamma_2)$ can be split into several sub-arcs of various types (see Figure 6): some are sub-arcs of γ_1 or γ_2, while the others are (adjacent to) sub-arcs of their symmetric images $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$. The conditioning on $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ ensures that the edges along the sub-arcs of the first type are open; the connections along the others depend on the exact explored configuration in a much more intricate way, but in any case the boundary condition imposed on the configuration inside $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$ is larger than the mixed boundary condition. Notice that *any* boundary condition dominates the free one and that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two sub-arcs of the first type (they are then wired). We deduce that the measure restricted to $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$ stochastically dominates the restriction of $\phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}$ to $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$.

From these observations, we deduce that for any increasing event B *depending only on edges in* $G_0(\gamma_1, \gamma_2)$,

$$\phi^{\mathrm{p}}_{p_{sd},q,m}(B | \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2) \geqslant \phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}(B). \tag{2.4}$$

In particular, we can apply this inequality to $\{\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\gamma_1, \gamma_2)\}$. Note that if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are connected in $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$, γ_1 and γ_2 are connected in $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$. The first event is of $\phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}$-probability at least $1/(1+q^2)$, implying

$$\begin{aligned} \phi^{\mathrm{p}}_{p_{sd},q,m}(\gamma_1 \leftrightarrow \gamma_2 | \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2) &\geqslant \phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}(\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\gamma_1, \gamma_2)) \\ &\geqslant \phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}(\tilde{\gamma}_1 \leftrightarrow \tilde{\gamma}_2) \geqslant \frac{1}{1+q^2}. \end{aligned} \tag{2.5}$$

Conclusion of the proof. Note the following obvious fact: if γ_1 and γ_2 intersect, the conditional probability that Γ_1 and Γ_2 intersect, knowing $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ is equal to 1 — in particular, it is greater than $1/(1+q^2)$. Now,

$$\begin{aligned} \phi^{\mathrm{p}}_{p_{sd},q,m}(\mathcal{C}_h(R)) &\geqslant \phi^{\mathrm{p}}_{p_{sd},q,m}(\mathcal{C}_h(R) \cap A) \\ &\geqslant \phi^{\mathrm{p}}_{p_{sd},q,m}(\{\Gamma_1 \leftrightarrow \Gamma_2\} \cap A) \\ &= \phi^{\mathrm{p}}_{p_{sd},q,m}\big(\phi^{\mathrm{p}}_{p_{sd},q,m}(\Gamma_1 \leftrightarrow \Gamma_2 | \Gamma_1, \Gamma_2) \mathbb{1}_A\big) \\ &\geqslant \frac{1}{1+q^2} \phi(A) \geqslant \frac{c(1)^3}{2(1+q)^2} \end{aligned}$$

where the first two inequalities are due to inclusion of events, the third one to the definition of conditional expectation, and the fourth and fifth ones, to (2.5) and (2.3). $\square$

An equivalent of Theorem 5 holds in the case of the infinite-volume random-cluster measure with wired boundary condition.

Corollary 9. *Let $\alpha > 1$ and $q \geqslant 1$; there exists $c(\alpha) > 0$ such that for every $n \geqslant 1$,*

$$\phi^1_{p_{sd},q}\big[\mathcal{C}_h\big([0,\alpha n)\times[0,n)\big)\big] \geqslant c(\alpha). \tag{2.6}$$

Proof. Let $\alpha > 1$ and $m > 2\alpha n > 0$. Using the invariance under translations of $\phi^{\mathrm{p}}_{p_{sd},q,m}$ and comparison between boundary conditions, we have

$$\phi^1_{p_{sd},q,[-\frac{m}{2},\frac{m}{2})^2}\big[\mathcal{C}_h\big([0,\alpha n)\times[0,n)\big)\big] \geqslant \phi^{\mathrm{p}}_{p_{sd},q,m}\big[\mathcal{C}_h\big([0,\alpha n)\times[0,n)\big)\big] \geqslant c(\alpha).$$

When m goes to infinity, the left hand side converges to the probability in infinite volume, so that

$$\phi^1_{p_{sd},q}\big[\mathcal{C}_h\big([0,\alpha n)\times[0,n)\big)\big] \geqslant c(\alpha). \qquad \square$$

Remark. The only place where we use the periodic and the (bulk) wired boundary conditions is in the estimate of Lemma 6. For instance, if one could prove that the probability for a square box to be crossed from top to bottom with free boundary conditions stays bounded away from 0 when n goes to infinity, then an equivalent of Theorem 5 would follow with free boundary conditions.

Uniform estimates with respect to boundary conditions should be true for $q \in [1,4)$; we expect the random-cluster model to be conformally invariant in the scaling limit. It should be false for $q \geqslant 4$. Indeed, for $q > 4$, the phase transition is (conjecturally) of first order in the sense that there should not be uniqueness of the infinite-volume measure. At $q = 4$, the random-cluster model should be conformally invariant, but the probability of a crossing with free boundary condition should converge to 0. Nevertheless, the probability that there is an open circuit surrounding the box of size n in the box of size $2n$ with free boundary condition should stay bounded away from 0.

Proving an equivalent of Theorem 5 with uniform estimates with respect to boundary conditions is an important question, since it would allow us to study the critical phase. The special case $q = 2$ has been derived recently in [10].

3 A sharp threshold theorem for crossing probabilities

The aim of this section is to understand the behavior of the function $p \mapsto \phi^{\xi}_{p,q,n}(A)$ for a non-trivial increasing event A. This increasing function is equal to 0 at $p = 0$ and to 1 at $p = 1$, and we are interested in the range of p for which its value is between ε and $1-\varepsilon$ for some positive ε (this range is usually referred to as a *window*). Under mild conditions on A, the window will be narrow for large graphs, and its width can be bounded above in terms of the size of the underlying graph, which is known as a *sharp threshold* behavior.

Historically, the general theory of sharp thresholds was first developed by Kahn, Kalai and Linial [21] (see also [13, 14, 22]) in the case of product measures. In lattice models such as percolation, these results are used via a differential equality known as Russo's formula, see [17, 28]. Both sharp threshold theory and Russo's formula were later extended to random-cluster measures with $q \geqslant 1$, see references below. These arguments being not totally standard, we remind the readers of the classic results we will employ and refer them to [18] for general results. Except for Theorem 12, the proofs are quite short so that it is natural to include them. The proofs are directly extracted from the Grimmett's monograph [18].

In the whole section, G will denote a finite graph; if e is an edge of G, let J_e be the random variable equal to 1 if the edge e is open, and 0 otherwise. We start with an example of a differential inequality, which will be useful in the proof of Theorem 2.

Proposition 10 (see [18, 19]). *Let $q \geqslant 1$; for any random-cluster measure $\phi^{\xi}_{p,q,G}$ with $p \in (0,1)$ and any increasing event A,*

$$\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) \geqslant 4\phi^{\xi}_{p,q,G}(A)\phi^{\xi}_{p,q,G}(H_A),$$

where $H_A(\omega)$ is the Hamming distance between ω and A.

Proof. Let A be an increasing event. The key step is the following inequality, see [4, 18], which can be obtained by differentiating with respect to p (for details of the computation, see Theorem (2.46) of [18]):

$$\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) = \frac{1}{p(1-p)}\sum_{e\in E}\left[\phi^{\xi}_{p,q,G}(\mathbb{1}_A J_e) - \phi^{\xi}_{p,q,G}(J_e)\phi^{\xi}_{p,q,G}(A)\right]. \tag{3.1}$$

A similar differential formula is actually true for any random variable X, but we will not use this fact in the proof. Define $|\eta|$ to be the number of open edges in the configuration, it is simply the sum of the random variables J_e, $e \in E$. With this notation, one can rewrite (3.1) as

$$\begin{aligned}\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) &= \frac{1}{p(1-p)}\left[\phi^{\xi}_{p,q,G}(|\eta|\mathbb{1}_A) - \phi^{\xi}_{p,q,G}(|\eta|)\phi^{\xi}_{p,q,G}(A)\right]\\ &= \frac{1}{p(1-p)}\Big[\phi^{\xi}_{p,q,G}\big((|\eta|+H_A)\mathbb{1}_A\big) - \phi^{\xi}_{p,q,G}\big(|\eta|+H_A\big)\phi^{\xi}_{p,q,G}(A)\\ &\qquad\qquad - \phi^{\xi}_{p,q,G}(H_A\mathbb{1}_A) + \phi^{\xi}_{p,q,G}(H_A)\phi^{\xi}_{p,q,G}(A)\Big]\\ &\geqslant \frac{1}{p(1-p)}\phi^{\xi}_{p,q,G}(H_A)\phi^{\xi}_{p,q,G}(A).\end{aligned}$$

To obtain the second line, we simply add and subtract the same quantity. In order to go from the second line to the third, we remark two things: in the second line, the third term equals 0 (when A occurs, the Hamming distance to A is 0), and the sum of the first two terms is positive thanks to the FKG inequality (indeed, it is easy to check that $|\eta|+H_A$ is increasing). The claim follows since $p(1-p) \leqslant 1/4$. □

This proposition has an interesting reformulation: integrating the formula between p_1 and $p_2 > p_1$, we obtain

$$\phi^{\xi}_{p_1,q,G}(A) \leqslant \phi^{\xi}_{p_2,q,G}(A)\,\mathrm{e}^{-4(p_2-p_1)\phi^{\xi}_{p_2,q,G}(H_A)} \tag{3.2}$$

(note that H_A is a decreasing random variable). If one can prove that the typical value of H_A is sufficiently large, for instance because A occurs with small probability, then one can obtain bounds for the probability of A. This kind of differential formula is very useful in order to prove the existence of a sharp threshold. The next example presents a sharper estimate of the derivative.

Intuitively, the derivative of $\phi^{\xi}_{p,q,G}(A)$ with respect to p is governed by the influence of one single edge, switching from closed to open (roughly speaking, considering the increasing coupling between p and $p + \mathrm{d}p$, it is unlikely that two edges switch their state). The following definition is therefore natural in this setting. The *(conditional) influence* on A of the edge $e \in E$, denoted by $I_A(e)$, is defined as

$$I_A(e) := \phi^{\xi}_{p,q,G}(A|J_e = 1) - \phi^{\xi}_{p,q,G}(A|J_e = 0).$$

Proposition 11. *Let $q \geqslant 1$ and $\varepsilon > 0$; there exists $c = c(q,\varepsilon) > 0$ such that for any random-cluster measure $\phi^{\xi}_{p,q,G}$ with $p \in [\varepsilon, 1-\varepsilon]$ and any increasing event A,*

$$\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) \geqslant c\sum_{e\in E} I_A(e).$$

Proof. Note that, by definition of $I_A(e)$,

$$\phi^{\xi}_{p,q,G}(\mathbb{1}_A J_e) - \phi^{\xi}_{p,q,G}(A)\phi^{\xi}_{p,q,G}(J_e) = I_A(e)\phi^{\xi}_{p,q,G}(J_e)\big(1-\phi^{\xi}_{p,q,G}(J_e)\big)$$

so that (3.1) becomes

$$\begin{aligned}\frac{d}{dp}\phi^{\xi}_{p,q,G}(A) &= \frac{1}{p(1-p)}\sum_{e\in E}\phi^{\xi}_{p,q,G}(J_e)\big(1-\phi^{\xi}_{p,q,G}(J_e)\big)I_A(e)\\ &= \sum_{e\in E}\frac{\phi^{\xi}_{p,q,G}(J_e)\big(1-\phi^{\xi}_{p,q,G}(J_e)\big)}{p(1-p)}I_A(e)\end{aligned}$$

from which the claim follows since the term

$$\frac{\phi^{\xi}_{p,q,G}(J_e)\big(1-\phi^{\xi}_{p,q,G}(J_e)\big)}{p(1-p)}$$

is bounded away from 0 uniformly in $p\in[\varepsilon,1-\varepsilon]$ and $e\in E$ when q is fixed. □

There has been an extensive study of the largest influence in the case of product measures. It was initiated in [21] and recently lead to important consequences in statistical models, see *e.g.* [5, 6]. The following theorem is a special case of the generalization to positively-correlated measures.

Theorem 12 (see [15]). *Let $q\geqslant 1$ and $\varepsilon>0$; there exists a constant $c=c(q,\varepsilon)\in(0,\infty)$ such that the following holds. Consider a random-cluster model on a graph G with $|E|$ denoting the number of edges of G. For every $p\in[\varepsilon,1-\varepsilon]$ and every increasing event A, there exists $e\in E$ such that*

$$I_A(e)\geqslant c\,\phi^{\xi}_{p,q,G}(A)\big(1-\phi^{\xi}_{p,q,G}(A)\big)\frac{\log|E|}{|E|}.$$

There is a particularly efficient way of using Proposition 11 together with Theorem 12. In the case of a translation-invariant event on a torus of size n, horizontal (resp. vertical) edges play a symmetric role, so that the influence is the same for all the edges of a given orientation. In particular, Proposition 11 together with Theorem 12 provide us with the following differential inequality:

Theorem 13. *Let $q\geqslant 1$ and $\varepsilon>0$. There exists a constant $c=c(q,\varepsilon)\in(0,\infty)$ such that the following holds. Let $n\geqslant 1$ and let A be a translation-invariant event on the torus of size n: for any $p\in[\varepsilon,1-\varepsilon]$,*

$$\frac{d}{dp}\phi^{\mathrm{p}}_{p,q,n}(A)\geqslant c\big(\phi^{\mathrm{p}}_{p,q,n}(A)(1-\phi^{\mathrm{p}}_{p,q,n}(A)\big)\log n.$$

For a non-empty increasing event A, we can integrate the previous inequality between two parameters $p_1<p_2$ (we recognize the derivative of $\log(x/(1-x))$) to obtain

$$\frac{1-\phi^{\mathrm{p}}_{p_1,q,n}(A)}{\phi^{\mathrm{p}}_{p_1,q,n}(A)}\geqslant\frac{1-\phi^{\mathrm{p}}_{p_2,q,n}(A)}{\phi^{\mathrm{p}}_{p_2,q,n}(A)}n^{c(p_2-p_1)}.$$

If we further assume that $\phi^{\xi}_{p_1,q,n}(A)$ stays bounded away from 0 uniformly in $n\geqslant 1$, we can find $c'>0$ such that

$$\phi^{\mathrm{p}}_{p_2,q,n}(A)\geqslant 1-c'n^{-c(p_2-p_1)}. \tag{3.3}$$

This inequality will be instrumental in the next section.

4 The proofs of Theorems 1 and 2

The previous two sections combine in order to provide estimates on crossing probabilities (see [5, 6] for applications in the case of percolation). Indeed, one can consider the event that *some* long rectangle is crossed in a torus. At $p = p_{sd}$, we know that the probability of this event is bounded away from 0 uniformly in the size of the torus (thanks to Theorem 5). Therefore, we can apply Theorem 13 to conclude that the probability goes to 1 when $p > p_{sd}$ (we also have an explicit estimate on the probability). It is then an easy step to deduce a lower bound for the probability of crossing a *particular* rectangle.

Theorem 1 is proved by constructing a path from 0 to infinity when $p > p_{sd}$, which is usually done by combining crossings of rectangles. There is a major difficulty in doing such a construction: one needs to transform estimates in the torus into estimates in the whole plane. One solution is to replace the periodic boundary condition by wired boundary condition. The path construction is a little tricky since it must propagate wired boundary conditions through the construction (see Proposition 16); it does not follow the standard lines.

Theorem 2 follows from a refinement of the previous construction in order to estimate the Hamming distance of a typical configuration to the event $\{0 \leftrightarrow \mathbb{L} \setminus [-n, n)^2\}$. It allows the use of Proposition 10, which improves bounds on the probability that the origin is connected to distance n. With these estimates, we show that the cluster size at the origin has finite moments of any order, whenever $p < p_{sd}$. Then, it is a standard step to obtain exponential decay in the sub-critical phase.

The following two lemmas will be useful in the proofs of both theorems. We start by proving that crossings of long rectangles exist with very high probability when $p > p_{sd}$.

Lemma 14. *Let $\alpha > 1$, $q \geqslant 1$ and $p > p_{sd}$; there exists $\varepsilon_0 = \varepsilon_0(p, q, \alpha) > 0$ and $c_0 = c_0(p, q, \alpha) > 0$ such that*

$$\phi^{\mathrm{p}}_{p,q,\alpha^2 n}\big(\mathcal{C}^v\big([0, n) \times [0, \alpha n)\big)\big) \geqslant 1 - c_0 n^{-\varepsilon_0} \tag{4.1}$$

for every $n \geqslant 1$.

Proof. The proof will make it clear that it is sufficient to treat the case of integer α, we therefore assume that α is a positive integer (not equal to 1). Let B be the event that there exists a vertical crossing of a rectangle with dimensions $(n/2, \alpha^2 n)$ in the torus of size $\alpha^2 n$. This event is invariant under translations and satisfies

$$\phi^{\mathrm{p}}_{p_{sd},q,\alpha^2 n}(B) \geqslant \phi^{\mathrm{p}}_{p_{sd},q,\alpha^2 n}\big(\mathcal{C}^v([0, n/2) \times [0, \alpha^2 n))\big) \geqslant c(2\alpha^2)$$

uniformly in n.

Let $p > p_{sd}$. Since B is increasing, we can apply Theorem 13 (more precisely (3.3)) to deduce that there exist $\varepsilon = \varepsilon(p, q, \alpha)$ and $c = c(p, q, \alpha)$ such that

$$\phi^{\mathrm{p}}_{p,q,\alpha^2 n}(B) \geqslant 1 - c n^{-\varepsilon}. \tag{4.2}$$

If B holds, one of the $2\alpha^3$ rectangles

$$[in/2, in/2 + n) \times [j\alpha n, (j + 1)\alpha n), \quad (i, j) \in \{0, \cdots, 2\alpha^2 - 1\} \times \{0, \cdots, \alpha - 1\}$$

must be crossed from top to bottom. We denote these events by A_{ij} — they are translates of $\mathcal{C}^v([0, n) \times [0, \alpha n))$. Using the FKG inequality in the second line (this is another instance of the "square-root trick" mentioned earlier), we find

$$\begin{aligned}\phi^{\mathrm{p}}_{p,q,\alpha^2 n}(B) &\leqslant 1 - \phi^{\mathrm{p}}_{p,q,\alpha^2 n}(B^c) \leqslant 1 - \phi^{\mathrm{p}}_{p,q,\alpha^2 n}(\cap_{i,j} A^c_{ij}) \\ &\leqslant 1 - \prod_{i,j} \phi^{\mathrm{p}}_{p,q,\alpha^2 n}(A^c_{ij}) = 1 - \Big[1 - \phi^{\mathrm{p}}_{p,q,\alpha^2 n}\big(\mathcal{C}^v([0, n) \times [0, \alpha n)\big)\Big]^{2\alpha^3}.\end{aligned}$$

Plugging (4.2) into the previous inequality, we deduce

$$\phi^{\mathrm{p}}_{p,q,\alpha^2 n}\left(\mathcal{C}^v\left([0,n)\times[0,\alpha n)\right)\right) \geqslant 1-(cn^{-\varepsilon})^{1/(2\alpha^3)}.$$

The claim follows by setting $c_0 := c^{1/(2\alpha)^3}$ and $\varepsilon_0 := \varepsilon/(2\alpha^3)$. □

Let $\alpha > 1$ and $n \geqslant 1$; we define the annulus

$$A_n^\alpha := [-\alpha^{n+1},\alpha^{n+1}]^2 \setminus [-\alpha^n,\alpha^n]^2.$$

An *open circuit* in an annulus is an open path which surrounds the origin. Denote by $\mathcal{A}_n^\alpha$ the event that there exists an open circuit surrounding the origin and contained in A_n^α, together with an open path from this circuit to the boundary of $[-\alpha^{n+2},\alpha^{n+2}]^2$, see Figure 7. The following lemma shows that the probability of $\mathcal{A}_n^\alpha$ goes to 1, provided that $p > p_{sd}$ and that we fixed wired boundary conditions on $[-\alpha^{n+2},\alpha^{n+2}]^2$.

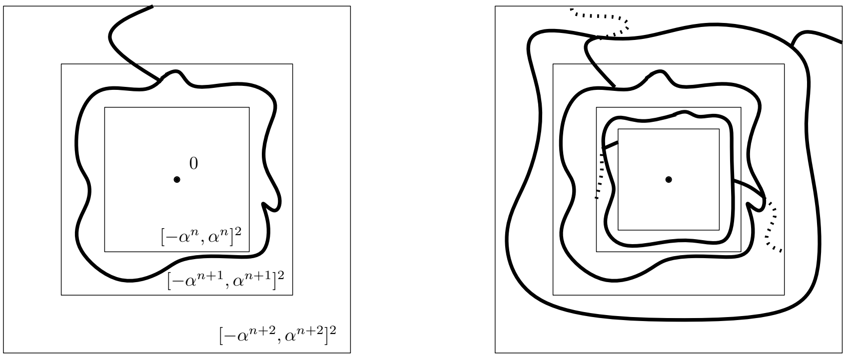

Figure 7: Left: The event $\mathcal{A}_n^\alpha$. **Right:** The combination of events $\mathcal{A}_n^\alpha$: we see that it indeed constructs a path from the origin to infinity.

Lemma 15. *Let $\alpha > 1$, $q \geqslant 1$ and $p > p_{sd}$; there exists $c_1 = c_1(p,q,\alpha)$ and $\varepsilon_1 = \varepsilon_1(p,q,\alpha)$ such that for every $n \geqslant 1$,*

$$\phi^1_{p,q,\alpha^{n+2}}(\mathcal{A}_n^\alpha) \geqslant 1 - c_1 e^{-\varepsilon_1 n}.$$

Proof. First, observe that $\mathcal{A}_n^\alpha$ occurs whenever the following events occur simultaneously:

- The following rectangles are crossed vertically:

$$R_1 := [\alpha^n,\alpha^{n+1}]\times[-\alpha^{n+1},\alpha^{n+1}],$$
$$R_2 := [-\alpha^{n+1},-\alpha^n]\times[-\alpha^{n+1},\alpha^{n+1}];$$

- The following rectangles are crossed horizontally:

$$R_3 := [-\alpha^{n+1},\alpha^{n+1}]\times[\alpha^n,\alpha^{n+1}],$$
$$R_4 := [-\alpha^{n+1},\alpha^{n+1}]\times[-\alpha^{n+1},-\alpha^n],$$
$$R_5 := [-\alpha^{n+2},\alpha^{n+2}]\times[-\alpha^{n+1},\alpha^{n+1}].$$

For the measure in the torus, these events have probability greater than $1 - c(\alpha^n)^{-\varepsilon}$ with $c = c_0(p,q,2\alpha/(\alpha-1))$ and $\varepsilon = \varepsilon_0(p,q,2\alpha/(\alpha-1))$. Using the FKG inequality, we obtain

$$\phi^{\mathrm{p}}_{p,q,\alpha^{n+2}}(\mathcal{A}_n^\alpha) \geqslant (1-c(\alpha^n)^{-\varepsilon})^5$$

from which we deduce the following estimate, harnessing the comparison between boundary conditions,

$$\phi^1_{p,q,\alpha^{n+2}}(\mathcal{A}_n^\alpha) \geqslant (1-c(\alpha^n)^{-\varepsilon})^5.$$

The claim follows by setting $c_1 := 5c$ and $\varepsilon_1 := \varepsilon\log\alpha$. □

The following proposition readily implies Theorem 1; It will also be useful in the proof of Theorem 2. We want to prove that the probability of the intersection of events $\mathcal{A}_n^\alpha$ is of positive probability when $p > p_{sd}$. So far, we know that there is an open circuit with very high probability when we consider the random-cluster measure with wired boundary condition in a slightly larger box. In order to prove the result, we assume the existence of a large circuit. Then, we iteratively condition on events $\mathcal{A}_{n-k}^\alpha$, $k \geqslant 0$. When conditioning 'from the outside to the inside', we guarantee that at step k, there exists an open circuit in A_{n-k+1}^α that surrounds A_{n-k}^α. Using comparison between boundary conditions, we can assure that the measure in A_{n-k}^α stochastically dominates the measure in A_{n-k+1}^α with wired boundary condition. In other words, we keep track of advantageous boundary conditions. Note that the reasoning, while reminiscent of Kesten's construction of an infinite path for percolation, is not standard.

Proposition 16. *Let $\alpha > 1$, $q \geqslant 1$ and $p > p_{sd}$; there exist $c, c_1, \varepsilon_1 > 0$ (depending on p, q and α) such that for every $N \geqslant 1$,*

$$\phi_{p,q}\left(\bigcap_{n\geqslant N} \mathcal{A}_N^\alpha\right) \geqslant c \prod_{k=N}^{\infty} (1 - c_1 \mathrm{e}^{-\varepsilon_1 k}) > 0.$$

Proof. Let $\alpha > 1$, $q \geqslant 1$, $p > p_{sd}$, $N \geqslant 1$ and recall that there is a unique infinite-volume measure $\phi_{p,q}$. For every $n \geqslant 1$, we know that

$$\phi_{p,q}\left(\bigcap_{k=N}^{n} \mathcal{A}_n^\alpha\right) = \phi_{p,q}(\mathcal{A}_n^\alpha) \prod_{k=N}^{n-1} \phi_{p,q}(\mathcal{A}_k^\alpha | \mathcal{A}_j^\alpha, k+1 \leqslant j \leqslant n). \tag{4.3}$$

On the one hand, let $k \in [N, n-1]$. Conditionally to $\mathcal{A}_j^\alpha$, $k+1 \leqslant j \leqslant n$, we know that there exists a circuit in the annulus A_{k+1}^α. Exploring from the outside, we shall consider the most exterior such circuit, denoted by Γ. Conditionally to $\Gamma = \gamma$, the unexplored part of the box $[-\alpha^{k+2}, \alpha^{k+2}]^2$ follows the law of a random-cluster configuration with wired boundary condition. In particular, the conditional probability that there exists a circuit in A_k^α connected to γ is greater than the probability that there exists a circuit in A_k^α connected to the boundary of $[-\alpha^{k+2}, \alpha^{k+2}]^2$ with wired boundary condition. Therefore, we obtain that almost surely

$$\begin{aligned}
\phi_{p,q}(\mathcal{A}_k^\alpha | \mathcal{A}_j^\alpha, k+1 \leqslant j \leqslant n) &= \phi_{p,q}\big(\phi_{p,q}(\mathcal{A}_k^\alpha | \Gamma = \gamma)\big) \\
&\geqslant \phi_{p,q}\big(\phi_{p,q,\alpha^{k+2}}^1(\mathcal{A}_k^\alpha)\big) \\
&\geqslant 1 - c_1 \mathrm{e}^{-\varepsilon_1 k}
\end{aligned}$$

where we have harnessed Lemma 15 in the last inequality.

On the other hand, for $p = p_{sd}$, consider the event $\mathcal{A}_n^\alpha$ in the bulk. Thanks to Corollary 9, its probability is bounded away from 0 uniformly in n. Since the event is increasing, we obtain that there exists $c = c(\alpha) > 0$ such that

$$\phi_{p,q}(\mathcal{A}_n^\alpha) = \phi_{p,q}^1(\mathcal{A}_n^\alpha) \geqslant c$$

for any $n \geqslant N$ and $p > p_{sd}$. Plugging the two estimates into (4.3), we obtain

$$\phi_{p,q}\left(\bigcap_{k=N}^{n} \mathcal{A}_n^\alpha\right) \geqslant c \prod_{k=N}^{n-1} (1 - c_1 \mathrm{e}^{-\varepsilon_1 k}) \geqslant c \prod_{k=N}^{\infty} (1 - c_1 \mathrm{e}^{-\varepsilon_1 k}).$$

Letting n go to infinity concludes the proof. $\square$

Proof of Theorem 1. The bound $p_c \geqslant p_{sd}$ is provided by Zhang's argument, as explained in Section 1. For $p > p_{sd}$, fix $\alpha > 1$. Applying Proposition 16 with $N = 1$, we find

$$\phi_{p,q}(0 \leftrightarrow \infty) \geqslant c\phi_{p,q}\left(\bigcap_{n\geqslant 1} \mathcal{A}_n^\alpha\right) > 0$$

so that p is super-critical. The constant $c > 0$ is due to the fact that we require $[-\alpha^2, \alpha^2]^2$ to contain open edges only ($c > 0$ exists using the finite energy property). Since p is super-critical for every $p > p_{sd}$, we deduce $p_c \leqslant p_{sd}$. $\square$

Proof of Theorem 2. Let x be a site of $\mathbb{Z}^2$, and let $\mathcal{C}_x$ be the cluster of x, *i.e.* the maximal connected component containing the site x. We denote by $|\mathcal{C}_x|$ its cardinality. We first prove that $|\mathcal{C}_x|$ has finite moments of any order. Then we deduce that the probability of $\{|\mathcal{C}_x| \geqslant n\}$ decays exponentially fast in n. The proof of the Step 2 is extracted from [18].

Step 1: finite moments for $|\mathcal{C}_x|$. Let $d > 0$ and $p < p_{sd}$; we want to prove that

$$\phi_{p,q}(|\mathcal{C}_x|^d) < \infty. \tag{4.4}$$

In order to do so, let $p_1 := (p + p_{sd})/2$ and define $D_n := \{x \leftrightarrow \mathbb{Z}^2 \setminus (x + [-n, n)^2)\}$; denote by H_n the Hamming distance to D_n. Note that H_n is the minimal number of closed edges that one must cross in order to go from x to the boundary of the box of size n centered at x. Let

$$\alpha := \exp\left[\frac{p_1 - p}{2d + 3}\right] > 1.$$

We know from Proposition 16, applied to the (super-critical) dual model, that the probability of $\bigcap_{n>N}(\mathcal{A}_n^\alpha)^\star$ is larger than $c\prod_N^\infty(1 - c_1e^{-\varepsilon_1 n}) > 0$ ($(\mathcal{A}_n^\alpha)^\star$ is the occurrence of $\mathcal{A}_n^\alpha$ in the dual model). Hence, there exists $N = N(p_1, q, \alpha)$ sufficiently large such that

$$\phi_{p_1,q}\left(\bigcap_{k\geqslant N}^\infty (\mathcal{A}_n^\alpha)^\star\right) \geqslant \frac{1}{2}.$$

On this event, H_n is greater than $(\log n/\log\alpha) - N$ since there is at least one closed circuit in each annulus A_k^α with $k \geqslant N$ (thus increasing the Hamming distance by 1). We obtain

$$\phi_{p_1,q}(H_n) \geqslant \left(\frac{\log n}{\log\alpha} - N\right)\phi_{p_1,q}\left(\bigcap_{k\geqslant N}^\infty (\mathcal{A}_n^\alpha)^\star\right) \geqslant \frac{\log n}{4\log\alpha}$$

for n sufficiently large. We can use (3.2) to find

$$\phi_{p,q}(D_n) \leqslant \phi_{p_1,q}(D_n)\exp\big[-4(p_1 - p)\phi_{p_1,q}(H_n)\big] \leqslant n^{-(2d+3)} \tag{4.5}$$

for n sufficiently large, from which (4.4) follows readily.

Step 2: exponential decay. Note that, from the first inequality of (4.5), it is sufficient to prove that for some constant $c > 0$,

$$\liminf_{n\to\infty} H_n/n \geqslant c \quad \text{a.s.}$$

in order to show that $\phi_{p,q}(D_n)$ decays exponentially fast.

Consider a (not necessarily open) self-avoiding path γ going from the origin to the boundary of the box of size n. We can bound from below the number $T(\gamma)$ of closed edges along this path by the following quantity:

$$\frac{T(\gamma)}{n} \geqslant \frac{1}{|\gamma|}T(\gamma) \geqslant \frac{1}{|\gamma|}\sum_{z\in\gamma}\frac{1}{|\mathcal{C}_z|} \geqslant \left(\frac{1}{|\gamma|}\sum_{z\in\gamma}|\mathcal{C}_z|\right)^{-1}.$$

Indeed, the number of closed edges in γ is larger than the number of distinct clusters intersecting γ. Moreover, if $\mathcal{C}$ denotes such a cluster, we have that $1 \geqslant \sum_{z\in\gamma}|\mathcal{C}|^{-1}\mathbb{1}_{z\in\mathcal{C}}$. The last inequality is due to Jensen's inequality. Since H_n can be rewritten as the infimum of $T(\gamma)$ on paths going from 0 to the boundary of the box, we obtain

$$\frac{H_n}{n} \geqslant \inf_{\gamma:0\leftrightarrow\mathbb{Z}^2\setminus\mathcal{B}_n}\left(\frac{1}{|\gamma|}\sum_{z\in\gamma}|\mathcal{C}_z|\right)^{-1}. \tag{4.6}$$

The goal of the end of the proof is to give an almost sure lower bound of the right-hand side. We will harness a two-dimensional analogue of the strong law of large number. In order to do that, we need to transform the random variables $|\mathcal{C}_z|$ to obtain independent variables. We start with the following domination.

Let $(\tilde{\mathcal{C}}_z)_{z\in\mathcal{B}_n}$ be a family of independent subsets of $\mathbb{Z}^2$ distributed as $\mathcal{C}_z$. We claim that $(|\mathcal{C}_z|)_{z\in\mathcal{B}_n}$ is stochastically dominated by the family $(M_z)_{z\in\mathcal{B}_n}$ defined as

$$M_z := \sup_{y\in\mathbb{Z}^2:z\in\tilde{\mathcal{C}}_y}|\tilde{\mathcal{C}}_y|.$$

Let $v_1, v_2, \ldots$ be a deterministic ordering of $\mathbb{Z}^2$. Given the random family $(\tilde{\mathcal{C}}_z)_{z\in\mathcal{B}_n}$, we shall construct a family $(D_z)_{z\in\mathcal{B}_n}$ having the same joint law as $(\mathcal{C}_z)_{z\in\mathcal{B}_n}$ and satisfying the following condition: for each z, there exists y such that $D_z \subset \tilde{\mathcal{C}}_y$. First, set $D_{v_1} = \tilde{C}_{v_1}$. Given D_{v_1}, D_{v_2}, $\ldots$, D_{v_n}, define $E = \bigcup_{i=1}^n D_{v_1}$. If $v_{n+1} \in E$, set $D_{v_{n+1}} = D_{v_j}$ for some j such that $v_{n+1} \in D_{v_j}$. If $v_{n+1} \notin E$, we proceed as follows. Let $\Delta_e E$ be the set of edges of $\mathbb{Z}^2$ having exactly one end-vertex in E. We may find a (random) subset F of $\tilde{C}_{v_{n+1}}$ such that F has the conditional law of C_{n+1} given that all edges in $\Delta_e E$ are closed; we now set $D_{v_{n+1}} = F$. We used the domain Markov property and the positive association. Indeed, we use that the law of $C_{v_{n+1}}$ depends only on $\Delta_e E$, and is stochastically dominated by the law of the cluster in the bulk without any conditioning. We obtain the required stochastic domination accordingly. In particular, $|\mathcal{C}_z| \leqslant M_z$ and M_z has finite moments.

From (4.6) and the previous stochastic domination, we get

$$\liminf_{n\to\infty}\frac{H_n}{n} \geqslant \liminf_{n\to\infty}\inf_{\gamma:0\leftrightarrow\mathbb{Z}^2\setminus\mathcal{B}_n}\left(\frac{1}{|\gamma|}\sum_{z\in\gamma}|\mathcal{C}_z|\right)^{-1} \geqslant \left(\limsup_{n\to\infty}\sup_{\gamma:0\leftrightarrow\mathbb{Z}^2\setminus\mathcal{B}_n}\frac{1}{|\gamma|}\sum_{z\in\gamma}M_z\right)^{-1}.$$

The second step is now to replace M_z by random variables that are independent. We can harness Lemma 2 of [11] to show that

$$\left(\limsup_{n\to\infty}\sup_{\gamma:0\leftrightarrow\mathbb{Z}^2\setminus\mathcal{B}_n}\frac{1}{|\gamma|}\sum_{z\in\gamma}M_z\right)^{-1} \geqslant \left(2\limsup_{n\to\infty}\sup_{|\Gamma|\geqslant n}\frac{1}{|\Gamma|}\sum_{z\in\gamma}|\tilde{\mathcal{C}}_z|^2\right)^{-1}$$

where the supremum is over all finite connected graphs Γ of cardinality larger than n that contain the origin (also called lattice animals).

Since the $|\tilde{\mathcal{C}}_z|^2$ are independent and have finite moments of any order, the main result of [9] guarantees that

$$2\limsup_{n\to\infty}\sup_{|\Gamma|\geqslant n}\frac{1}{|\Gamma|}\sum_{z\in\gamma}|\tilde{\mathcal{C}}_z|^2\leqslant C\quad a.s.$$

for some $C > 0$. Therefore, with positive probability, $\liminf H_n/n$ is greater than a given constant, which concludes the proof. □

5 The critical point for the triangular and hexagonal lattices

Let $\mathbb{T}$ be the triangular lattice of mesh size 1, embedded in the plane in such a way that the origin is a vertex and the edges of $\mathbb{T}$ are parallel to the lines of equations $y = 0$, $y = \sqrt{3}x/2$ and $y = -\sqrt{3}x/2$. The dual graph of this lattice is a hexagonal lattice, denoted by $\mathbb{H}$, see Figure 8. Via planar duality, it is sufficient to handle the case of the triangular lattice in order to prove Theorem 4. Define $p_{\mathbb{T}}$ as being the unique $p \in (0, 1)$ such that $y^3 + 3y^2 - q = 0$, where $y := p_{\mathbb{T}}/(1 - p_{\mathbb{T}})$. The goal is to prove that $p_c(\mathbb{T}) = p_{\mathbb{T}}$.

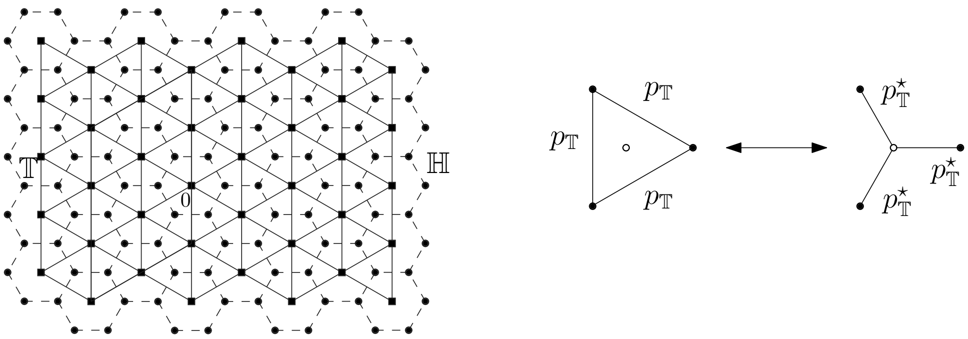

Figure 8: Left: The triangular lattice $\mathbb{T}$ with its dual lattice $\mathbb{H}$. **Right:** The exchange of the two patterns does not alter the random-cluster connective properties of the black vertices.

The general strategy is the same as in the square lattice case. We prove that at $p = p_{\mathbb{T}}$, a crossing estimate similar to Theorem 5 holds. Sharp threshold arguments and proofs of Section 4 can be adapted *mutatis mutandis*, replacing square-shaped annuli by hexagonal-shaped annuli. The crossing estimate must be slightly modified, and we present the few changes. It harnesses the planar-duality between the triangular and the hexagonal lattices, and the so-called *star-triangle transformation* (see *e.g.* Section 6.6 of [18] and Figure 8). We assume that the reader is already familiar with the star-triangle transformation.

Let $e_1 = \sqrt{3}/2 + \mathrm{i}/2$ and $e_2 = \mathrm{i}$; whenever we write coordinates, they are understood as referring to the basis (e_1, e_2). A 'rectangle' $[a, b) \times [c, d)$ is the set of points in $z \in \mathbb{T}$ such that $z = \lambda e_1 + \mu e_2$ with $\lambda \in [a, b)$ and $\mu \in [c, d)$ (it has a lozenge shape, see *e.g.* Figure 10). By analogy with the case of the square lattice, $\mathcal{C}_v(D)$ denotes the event that there exists a path between the top and the bottom sides of D which stays inside D. Such a path is called a *vertical open crossing* of the rectangle. Other quantities are defined similarly. Let $\mathbb{T}_m$ be the torus of size m constructed using the "rectangle" $[0, m] \times [0, m]$ with respect to the basis (e_1, e_2). We present the crossing estimate in the case of the torus $\mathbb{T}_m$ (deriving the bulk estimate follows the same lines as in the square lattice case); $\phi^{\mathrm{p}}_{p_{sd},q,m}$ denotes the random-cluster measure on $\mathbb{T}_m$.

Theorem 17. *Let $\alpha > 1$ and $q \geqslant 1$. There exists $c(\alpha) > 0$ such that for every $m > \alpha n > 0$,*

$$\phi^{\mathrm{p}}_{p_{\mathbb{T}},q,m}\big(\mathcal{C}_h([0, n) \times [0, \alpha n))\big) \geqslant c(\alpha). \tag{5.1}$$

The main difficulty is the adaptation of Lemma 7. Define the line $d := -\sqrt{3}/3 + \mathrm{i}\mathbb{R}$. The orthogonal symmetry σ_d with respect to d maps $\mathbb{T}$ to another triangular lattice. Note that this lattice is a sub-lattice of $\mathbb{H}$ (in the sense that its vertices are also vertices of $\mathbb{H}$). Let γ_1 and γ_2 be two paths satisfying the following Hypothesis ($\star$), see Figure 9:

- γ_1 remains on the left of d and γ_2 remains on the right,
- γ_2 begins at 0 and γ_1 begins on a site of $\mathbb{T} \cap (-\sqrt{3}/2 + \mathrm{i}\mathbb{R}_+)$,
- γ_1 and $\sigma_d(\gamma_2)$ do not intersect (as curves in the plane),
- γ_1 and $\sigma_d(\gamma_2)$ end at two sites (one primal and one dual) which are at distance $\sqrt{3}/3$ from one another.

When following the paths in counter-clockwise order, we can create a circuit by linking the end points of γ_1 and $\sigma_d(\gamma_2)$ by a straight line, the start points of $\sigma_d(\gamma_2)$ and γ_2, the end points of γ_2 and $\sigma_d(\gamma_1)$, and the start points of $\sigma_d(\gamma_1)$ and γ_1. The circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$ surrounds a set of vertices of $\mathbb{T}$. Define the graph $G(\gamma_1, \gamma_2)$ with sites being site of $\mathbb{T}$ that are surrounded by the circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$, and with edges of $\mathbb{T}$ that remain entirely inside the circuit (boundary included).

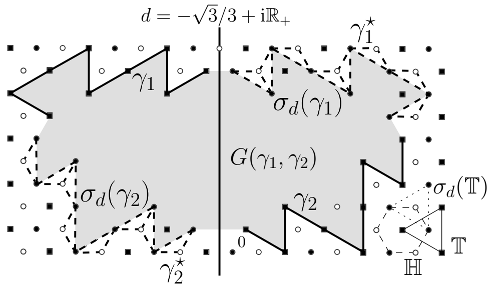

Figure 9: The graph $G(\gamma_1, \gamma_2)$ with the two solid arcs γ_1 and γ_2 and the dashed arcs $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$. The dual arcs $\gamma_1^\star$ and $\gamma_2^\star$ are dotted.

We will need an additional technical condition, which we present now. Note that for any edge of $\sigma_d(\mathbb{T})$ there is one vertex of $\mathbb{T}$ and one vertex of $\mathbb{H}$ at distance $\sqrt{3}/6$ from its midpoint. We assume that for any edge of $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$, the associated vertex of $\mathbb{T}$ is in the interior of the domain $G(\gamma_1, \gamma_2)$ (therefore, the associated vertex of $\mathbb{H}$ is outside the domain, see white vertices in Fig 9). We will refer to this condition as Hypothesis ($\star\star$).

The *mixed boundary condition* on this graph is wired on γ_1 (all the edges are pairwise connected), wired on γ_2, and free elsewhere. We denote the measure on $G(\gamma_1, \gamma_2)$ with parameters $(p_{\mathbb{T}}, q)$ and mixed boundary condition by $\phi_{p_{\mathbb{T}},q,\gamma_1,\gamma_2}$ or more simply ϕ_{γ_1,γ_2}. With these definitions, we have an equivalent of Lemma 7:

Lemma 18. *For any γ_1, γ_2 satisfying Hypotheses ($\star$) and ($\star\star$), we have*

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geqslant \frac{1}{1+q^2}.$$

Proof. As previously, if γ_1 and γ_2 are not connected, $\gamma_1^\star$ and $\gamma_2^\star$ are connected in the dual model, where $\gamma_1^\star, \gamma_2^\star \subset \mathbb{H}$ are the dual arcs bordering $G(\gamma_1, \gamma_2)$ close to $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$. Thanks to Hypothesis ($\star\star$) and the mixed boundary condition, this event is equivalent to the event that $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ are dual connected. Using Hypothesis ($\star$) and the symmetry, we deduce

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + \sigma_d * \phi^\star_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) = 1,$$

where as before $\sigma_d * \phi^\star_{\gamma_1,\gamma_2}$ denotes the push-forward under the symmetry σ_d of the dual measure of ϕ_{γ_1,γ_2} — in particular, it lies on $\sigma_d(\mathbb{H})$ and the edge-weight is $p^\star_{\mathbb{T}}$. This lattice contains the sites of $\mathbb{T}$ and those of another copy of the triangular lattice which we will denote by $\mathbb{T}'$. Since γ_1 and γ_2 are two paths of $\mathbb{T}$, one can use the star-triangle transformation for any triangle of $\mathbb{T}$ included in $G(\gamma_1,\gamma_2)$ that contains a vertex of $\mathbb{T}'$: one obtains that $\sigma_d * \phi^\star_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2)$ is equal to the probability of γ_1 and γ_2 being connected, in a model on $\mathbb{T}$ with edge-weight $p_{\mathbb{T}}$. Here, we need Hypothesis ($\star\star$) again in order to ensure that all the triangles containing a vertex of $\mathbb{T}'$ have no edges on the boundary (which would have forbidden the use of the star-triangle transformation). The same observation as in the case of the square lattice shows that the boundary conditions are the same as for ϕ_{γ_1,γ_2}, except that arcs γ_1 and γ_2 are wired together. The same reasoning as in Lemma 7 implies that

$$\sigma_d * \phi^\star_{\gamma_1,\gamma_2}\big(\gamma_1 \leftrightarrow \gamma_2\big) \leqslant q^2 \phi_{\gamma_1,\gamma_2}\big(\gamma_1 \leftrightarrow \gamma_2\big),$$

and the claim follows readily. $\square$

The existence of $c(1)$ is obtained in the same way as in the case of the square lattice, with only the obvious modifications needed; we leave the details as an "exercise for the reader". Theorem 17 is derived exactly as in Section 2, as soon as an equivalent of Proposition 8 holds:

Proposition 19. *There exists a constant $c(3/2) > 0$ such that, for all $m > 3n/2 > 0$,*

$$\phi^{\mathrm{p}}_{p_{\mathbb{T}},q,m}\big(\mathcal{C}^v([0,3n/2) \times [0,n))\big) \geqslant c(3/2).$$

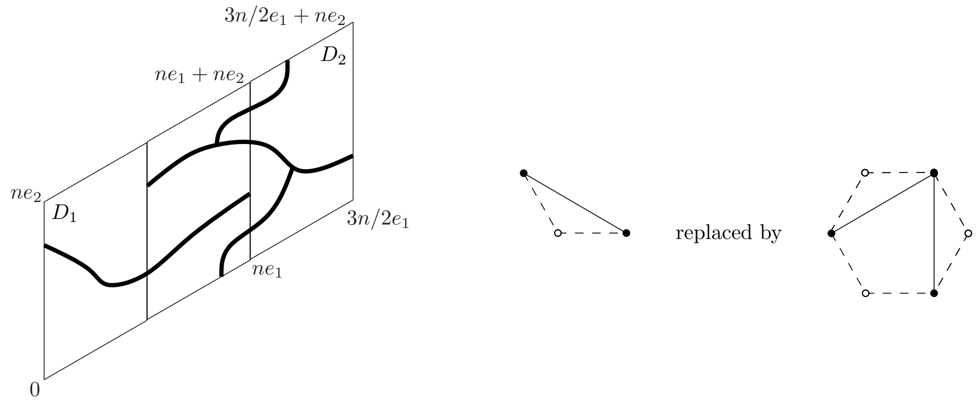

Figure 10: Left: The set $[0,3n/2) \times [0,n)$ and the event A. **Right:** One can obtain the path Γ_1' from Γ_1 by replacing any bad edge with two edges. Since Γ_1 is the top-most crossing, it contains no double edges and this construction can be done.

Proof. The general framework of the proof is the same as before, but some technicalities occur because the underlying lattice is not self-dual. Consider the rectangle $D = [0,3n/2) \times [0,n)$, which is the union of rectangles $D_1 = [0,n) \times [0,n)$ and $D_2 = [n/2,3n/2) \times [0,n)$, see Figure 10. Let A be the event that:

- D_1 and D_2 are both crossed horizontally (each crossing has probability at least $c(1)$ to occur);
- $[n/2,n) \times \{0\}$ (resp. $[n,3n/2) \times \{n\}$) is connected inside D_2 to the top side (resp. to the bottom). Using the FKG inequality and symmetries of the lattice, this event occurs with probability larger than $c(1)^2/4$.

Therefore, A has probability larger than $c(1)^4/4$.

When A occurs, define Γ_1 to be the top-most crossing of the rectangle D_1, and Γ_2 the right-most crossing in D_2 between $[n/2, n) \times \{0\}$ and the top side of D_2. Note that Γ_2 is automatically connecting $[n/2, n) \times \{0\}$ to the right edge and to $[n, 3n/2) \times \{n\}$. In order to conclude, it is sufficient to prove that Γ_1 and Γ_2 are connected with probability larger than some positive constant.

Consider the lowest path Γ_1' above Γ_1 which satisfies the following property: for any edge e in Γ_1', the associated site of $\sigma_d(\mathbb{H})$ (see the definition of Hypothesis ($\star\star$)) is in the connected component of $D_1 \setminus \Gamma_1'$ *above* Γ_1'. Such a path can be obtained from Γ_1 by replacing every 'bad' edge with the other two edges of a triangle, as shown in Figure 10. Since Γ_1 is the top-most crossing, it cannot have double edges and the path Γ_1' can be constructed. In particular it ends at the same point as Γ_1, and it goes from left to right. Note that it is not necessarily open. We define Γ_2' similarly in the obvious way (the left-most path on the right of Γ_2 such that for any edge of Γ_2', the associate site of $\sigma_d(\mathbb{H})$ is on the right of Γ_2').

We now sketch the end of the proof. Apply a construction similar to the proof of Proposition 8 in order to create a domain $G(\Gamma_1', \Gamma_2')$. With mixed boundary conditions, the probability of connecting Γ_1' to Γ_2' in $G(\Gamma_1', \Gamma_2')$ is larger than $1/(1+q^2)$ (Γ_1' and Γ_2' have been constructed in such a way that Hypothesis ($\star\star$) is fulfilled). But Γ_1 disconnects Γ_1' from Γ_2', and Γ_2 disconnects Γ_2' from Γ_1. Using boundary conditions inherited from the fact that Γ_1 and Γ_2 are crossings, one can prove that Γ_1 is connected to Γ_2 in $G(\Gamma_1', \Gamma_2')$ with probability larger than $1/(1+q^2)$. The end of the proof follows exactly the same lines as in the case of the square lattice. □

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