BRIDGES AND RANDOM TRUNCATIONS OF RANDOM MATRICES

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ABSTRACT. Let U be a Haar distributed matrix in $\mathbb{U}(n)$ or $\mathbb{O}(n)$. In a previous paper, we proved that after centering, the two-parameter process

$$T^{(n)}(s,t) = \sum_{i \leq \lfloor ns \rfloor, j \leq \lfloor nt \rfloor} |U_{ij}|^2 \ , \ s,t \in [0,1]$$

converges in distribution to the bivariate tied-down Brownian bridge. In the present paper, we replace the deterministic truncation of U by a random one, in which each row (resp. column) is chosen with probability s (resp. t) independently. We prove that the corresponding two-parameter process, after centering and normalization by $n^{-1/2}$ converges to a Gaussian process. On the way we meet other interesting convergences.

1. Introduction

Let us consider a unitary matrix U of size $n \times n$ and two fixed integers $p < n$ and $q < n$. Let us call $U^{p,q}$ the (rectangular) matrix obtained by deleting the last $n-p$ rows and $n-q$ columns from U. It is well known that if U is Haar distributed in $\mathbb{U}(n)$, the random matrix $U^{p,q}\,(U^{p,q})^*$ has a Jacobi matricial distribution and that if p, q and $n \to \infty$ with $(p/n, q/n) \to (s,t) \in (0,1)^2$, its empirical spectral distribution converges to a limit $\mathcal{D}_{s,t}$ (see for instance [10]), often called the generalized Kesten-McKay distribution.

In [14] we studied the trace of $U^{p,q}\,(U^{p,q})^*$ which is also the square of the Frobenius (or Euclidean) norm of $U^{p,q}$. Actually we set $p = \lfloor ns \rfloor, q = \lfloor nt \rfloor$ and considered the process indexed by $s,t \in [0,1]$. We proved that, after centering, but without any normalization, the process converges in distribution, as $n \to \infty$, to a bivariate tied-down Brownian bridge. Previously, Chapuy [9] proved a similar result for permutation matrices, with an $n^{-1/2}$ normalization.

Besides, for purposes of random geometry analysis, Farrell has proposed another model in [18] (see also [17]), deleting randomly and independently a proportion $1-s$ of rows and a proportion $1-t$ of columns from a Haar distributed matrix in $\mathbb{U}(n)$. If $\mathcal{U}^{s,t}$ denotes the matrix so obtained, he proved

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that (for fixed s,t) the empirical spectral distribution of $\mathcal{U}^{s,t}\left(\mathcal{U}^{s,t}\right)^*$ converges again, as $n\to\infty$, to $\mathcal{D}_{s,t}$.

It is then tempting to study the trace of $\mathcal{U}^{s,t}\left(\mathcal{U}^{s,t}\right)^*$ as a process, after having defined a probability space where all random truncations live simultaneously. For that purpose we define a double array of $2n$ auxiliary independent uniform variables $R_1,\dots,R_n,C_1,\dots,C_n$ and then, for any choice of (s,t), obtain the matrix $\mathcal{U}^{s,t}$ by removing from U rows with indices not in $\{i : R_i\le s\}$ and columns with indices not in $\{j : R_j\le t\}$. This gives us a coupled realization of the $\mathcal{U}^{s,t}$, reminiscent of the "standard coupling" for percolation models. Then, we notice that in the first model, the invariance of the Haar distribution on $\mathbb{U}(n)$ implies that we could have deleted any fixed set of $n-p$ rows and $n-q$ columns. So, we can consider the random truncation model as the result of the subordination of the deterministic truncation model by a couple of binomial processes. In other words, we treat the latter uniform variables as an environment, and state quenched and annealed convergences.

For instance, we will prove that after convenient centering and without normalization, the above process converges (quenched) to a bivariate Brownian bridge, but that after another centering and with normalization by $n^{-1/2}$ it converges (annealed) to a Gaussian process which is no more a bivariate Brownian bridge.

We use the space $D([0,1]^2)$ endowed with the topology of Skorokhod (see [5]). It consists of functions from $[0,1]^2$ to $\mathbb{R}$ which are at each point right continuous (with respect to the natural partial order of $[0,1]^2$) and admit limits in all "orthants". For the sake of completeness, we treat also the one-parameter process, i.e. truncation of the first column of the unitary matrix, and the case of permutation matrices.

Actually, Farrell considered first the (deterministic) discrete Fourier transform (DFT) matrix

$$F_{jk}=\frac{1}{\sqrt{n}}e^{-2i\pi(j-1)(k-1)/n}\;,\quad j,k=1,\dots,n\,, \tag{1.1}$$

and proved that after random truncation, a Haar unitary matrix has the same limiting singular value distribution. In a still more recent paper ([1]), Anderson and Farrell explain the connection with liberating sequences. In some sense, the randomness coming from the truncation is stronger than the randomness of the initial matrix. Here, we have considered also the DFT matrix, but we can as well consider any (random or not random) matrix whose elements are all of modulus $n^{-1/2}$, for instance a (normalized) complex Hadamard matrix.

The paper is organized as follows. In Sec. 2 we provide some definitions. Section 3 is devoted to our main results, the convergence of one-parameter (Theorem 3.2) and two-parameter processes (Theorems 3.5 and 3.8). In Sec. 4, we introduce the subordination method, which allows to give the proofs of the latter theorems as examples of application. In Section 5, we go back to the direct method, used in ([15]) which does not assume that the result of deterministic truncation is known. This point of view leads to conjectures.

2. Notation

We introduce the random processes that we will consider in this paper and the various limiting processes involved.

2.1. **The main statistics.** Let $\mathbb{U}(n)$ (resp. $\mathbb{O}(n)$) be the group of unitary (resp. orthogonal) $n \times n$ matrices and $U = (U_{ij})$ its generic element. We equip $\mathbb{U}(n)$ (resp. $\mathbb{O}(n)$) with the Haar probability measure $\pi^{(n)}(dU)$.
To define two systems of projective Bernoulli choices of rows and columns, we will need two independent families of independent random variables uniformly distributed on $[0,1]$ so that we can treat the randomness coming from the truncation as an environment. More specifically, the space of environments is $\Omega = [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$, whose generic element is denoted by $\omega = (R_i, i \geq 1, C_j, j \geq 1)$. We equip Ω with the probability measure $d\omega$ which is the infinite product of copies of the uniform distribution. In the sequel, "for almost every ω" will mean "for $d\omega$ - almost every ω."

For the one-parameter model, we introduce two processes with values in $D([0,1])$:

$$B_s^{(n)}(U) = \sum_1^{\lfloor ns \rfloor} |U_{i1}|^2 \,, \tag{2.1}$$

$$\mathcal{B}_s^{(n)}(\omega, U) = \sum_1^{n} |U_{i1}|^2 1_{R_i \leq s} \,. \tag{2.2}$$

For the two-parameter model, we introduce processes with values in $D([0,1]^2)$:

(1) $T^{(n)}(U)$ defined by

$$T_{s,t}^{(n)}(U) = \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} |U_{ij}|^2 \,,$$

(2) $\mathcal{T}^{(n)}(\omega, U)$ defined by

$$\mathcal{T}_{s,t}^{(n)}(\omega, U) = \sum_{i=1}^{n} \sum_{j=1}^{n} |U_{ij}|^2 \mathbf{1}_{R_i \leq s} \mathbf{1}_{C_j \leq t} \,. \tag{2.3}$$

The counting processes $S^{(n)}$ and $S'^{(n)}$ are defined by

$$S_s^{(n)}(\omega) = \sum_{i=1}^{n} \mathbf{1}_{R_i \leq s} \quad , \quad S_t'^{(n)}(\omega) = \sum_{j=1}^{n} \mathbf{1}_{C_j \leq t} \,, \tag{2.4}$$

and their normalized version $\widetilde{S}^{(n)}$ and $\widetilde{S}'^{(n)}$ by

$$\widetilde{S}_s^{(n)} = n^{-1/2} \left(S_s^{(n)} - ns \right) \quad , \quad \widetilde{S}_t'^{(n)} = n^{-1/2} \left(S_t'^{(n)} - nt \right) \,. \tag{2.5}$$

2.2. **Gaussian processes and bridges.** The classical Brownian bridge denoted by B_0 is a centered Gaussian process with continuous paths defined on $[0,1]$, of covariance

$$\mathbb{E}\left(B_0(s)B_0(s')\right) = s \wedge s' - ss' .$$

The bivariate Brownian bridge denoted by $B_{0,0}$ is a centered Gaussian process with continuous paths defined on $[0,1]^2$ of covariance

$$\mathbb{E}\left(B_{0,0}(s,t)B_{0,0}(s',t')\right) = (s \wedge s')(t \wedge t') - ss'tt'.$$

The tied-down bivariate Brownian bridge denoted by $W^{(\infty)}$ is a centered Gaussian process with continuous paths defined on $[0,1]^2$ of covariance

$$\mathbb{E}[W^{(\infty)}(s,t)W^{(\infty)}(s',t')] = (s \wedge s' - ss')(t \wedge t' - tt').$$

Let also $\mathcal{W}^{(\infty)}$ be the centered Gaussian process with continuous paths defined on $[0,1]^2$ of covariance

$$\mathbb{E}[\mathcal{W}^{(\infty)}(s,t)\mathcal{W}^{(\infty)}(s',t')] = ss'(t \wedge t') + (s \wedge s')tt' - 2ss'tt'.$$

It can be defined also as

$$\mathcal{W}^{(\infty)}(s,t) = sB_0(t) + tB_0'(s) \tag{2.6}$$

where B_0 and B_0' two independent one-parameter Brownian bridges.

At last we will meet the process denoted by $B_0 \otimes B_0$ which is a centered process with continuous paths defined on $[0,1]^2$ by

$$B_0 \otimes B_0(s,t) = B_0(s)B_0'(t)$$

where B_0 and B_0' are two independent Brownian bridges. This process is not Gaussian, but it has the same covariance as $W^{(\infty)}$.

Similarly, if F and G are two processes with values in $D([0,1])$, defined on the same probability space, we denote by $F \otimes G$ the process with values in $D([0,1]^2)$ defined by

$$F \otimes G(s,t) = F(s)G(t).$$

For simplicity we denote by I the deterministic trivial process $I_s = s$.

3. Convergence in distribution

We present unified results in the cases of the unitary and orthogonal groups. For this purpose we use the classical notation

$$\beta' = \frac{\beta}{2} = \begin{cases} 1/2 & \text{in the orthogonal case,} \\ 1 & \text{in the unitary case.} \end{cases}$$

3.1. **One-parameter processes.** Let us begin with the one-parameter processes, where $\xrightarrow{\text{law}}$ means convergence in distribution in $D([0,1))$. We present successively the results for the deterministic and random truncations.

Lemma 3.1. *Under* $\pi^{(n)}$,

$$n^{1/2}\left(B^{(n)} - I\right) \xrightarrow{\text{law}} \sqrt{\beta'^{-1}} B_0\,. \tag{3.1}$$

This convergence (3.1) is well known since at least Silverstein [23] (in the case $\beta' = 1$). It can be viewed as a direct consequence of the fact that the vector $(|U_{i1}|^2, i = 1, \dots, n)$ follows the Dirichlet $(\beta', \dots, \beta')$ distribution on the simplex.

Theorem 3.2. (1) *(Quenched) For almost every* ω*, the push-forward of* $\pi^{(n)}(dU)$ *by the map*

$$U \mapsto n^{1/2}\left(\mathcal{B}^{(n)}(\omega, U) - n^{-1}S^{(n)}(\omega)\right) \tag{3.2}$$

converges weakly to the distribution of $\sqrt{\beta'^{-1}} B_0$.

(2) *(Annealed) Under the joint probability measure* $d\omega \otimes \pi^{(n)}(dU)$

$$n^{1/2}\left(\mathcal{B}^{(n)} - I\right) \xrightarrow{\text{law}} \sqrt{1 + \beta'^{-1}} B_0. \tag{3.3}$$

3.2. **Two-parameter processes.** Let us continue with the two-parameter processes, where now $\xrightarrow{\text{law}}$ means convergence in distribution in $D([0,1]^2)$. We study three models. In the first one, U is the DFT matrix defined in (1.1). In the second one, U is sampled from the Haar measure on $\mathbb{U}(n)$ or $\mathbb{O}(n)$. Though the proof is much more involved than in the first model, the annealed convergence gives the same limit. At last, for the sake of completeness, we state here a result when U is chosen uniformly among $n \times n$ permutation matrices.

3.2.1. *DFT.* Here, there is no randomness in U, so that we have the decomposition:

$$\mathcal{T}^{(n)} = \frac{S^{(n)} \otimes S'^{(n)}}{n} = nI \otimes I + \widetilde{S}^{(n)} \otimes \widetilde{S'}^{(n)} + n^{1/2}\left(\widetilde{S}^{(n)} \otimes I + I \otimes \widetilde{S'}^{(n)}\right) \tag{3.4}$$

Theorem 3.3. *If* U *is the DFT matrix (or more generally if* U *is any matrix such that* $|U_{ij}|^2 = 1/n$ *a.s. for every* i, j*), then under the probability measure* $d\omega$

$$n^{-1/2}\left(\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)}\right) \xrightarrow{\text{law}} \mathcal{W}^{(\infty)}\,. \tag{3.5}$$

Proof. It is straightforward since the processes $S^{(n)}$ and $S'^{(n)}$ are independent. From (3.4) we have successively

$$\mathbb{E}\mathcal{T}^{(n)} = n^{-1}\mathbb{E}\left(S^{(n)} \otimes S'^{(n)}\right) = nI \otimes I,$$

$$n^{-1/2}\left(\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)}\right) = n^{-1/2}\left(\widetilde{S}^{(n)} \otimes \widetilde{S'}^{(n)}\right) + \widetilde{S}^{(n)} \otimes I + I \otimes \widetilde{S'}^{(n)}\,.$$

Applying Donsker's theorem, we get

$$(\widetilde{S}^{(n)}, \widetilde{S'}^{(n)}) \xrightarrow{\text{law}} (B_0, B'_0)$$

so that

$$n^{-1/2}\left(\widetilde{S}^{(n)} \otimes \widetilde{S'}^{(n)}\right) \to 0$$

in probability and

$$S^{(n)} \otimes I + I \otimes \widetilde{S'}^{(n)} \xrightarrow{\text{law}} B_0 \otimes I + I \otimes B'_0 = \mathcal{W}^{(\infty)} .$$

□

3.2.2. *Haar unitary or orthogonal matrices.* The case of deterministic truncation was treated in our previous paper and recalled now.

Theorem 3.4 ([14])**.** *Under $\pi^{(n)}$,*

$$W^{(n)} := T^{(n)} - \mathbb{E}T^{(n)} \xrightarrow{\text{law}} \sqrt{\beta'^{-1}}W^{(\infty)} . \tag{3.6}$$

The case of random truncation is ruled by the following theorem, which is the main result of the present paper.

Theorem 3.5. (1) *(Quenched) For almost every ω, the push-forward of $\pi^{(n)}(dU)$ on $D([0,1])^2$ by the mapping*

$$U \mapsto \mathcal{V}^{(n)} := \mathcal{T}^{(n)}(\omega, U) - \frac{S^{(n)}(\omega) \otimes S'^{(n)}(\omega)}{n} \tag{3.7}$$

converges weakly to the distribution of $\sqrt{\beta'^{-1}}W^{(\infty)}$.

(2) *(Annealed) Under the joint probability measure $d\omega \otimes \pi^{(n)}(dU)$,*

$$n^{-1/2}\left(\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)}\right) \xrightarrow{\text{law}} \mathcal{W}^{(\infty)} .$$

Remark 3.6. *Let $M^{p,q} = U^{p,q}\left(U^{p,q}\right)^*$ and $\mathcal{M}^{s,t} = \mathcal{U}^{s,t}\left(\mathcal{U}^{s,t}\right)^*$. For s,t fixed, the random variables $T^{(n)}_{s,t}$ and $\mathcal{T}^{(n)}_{s,t}$ are linear functionals of the empirical spectral distribution of $M^{\lfloor ns\rfloor,\lfloor nt\rfloor}$ and $\mathcal{M}^{s,t}$ respectively. For classical models in Random Matrix Theory, the convergence of fluctuations of such linear functionals do not need a normalizing factor, since the variance is bounded (the eigenvalues are repelling each other). Here, this is indeed the case for $T^{(n)}_{s,t}$ (see* [16] *for the complete behavior for general tests functions). But, in the case of $\mathcal{T}^{(n)}_{s,t}$, we have* $\mathrm{Var}\left(\mathbb{E}[\mathcal{T}^{(n)}_{s,t}|\omega]\right) = O(n)$*, which demands a normalization. Notice however that the main source of this variance lies in the fluctuations of the number of columns and lines removed from the initial matrix, rather than in the matrix itself.*

3.2.3. *Permutation matrices.* Let us call $p^{(n)}$ the uniform measure on the group $\mathcal{S}_n$ of permutation matrices of $\{1,\dots,n\}$. The deterministic truncation was treated by Chapuy.

Theorem 3.7 ([9]). *Under $p^{(n)}$ we have*

$$n^{-1/2}\left(T^{(n)} - \mathbb{E}T^{(n)}\right) \quad \xrightarrow{\text{law}} \quad W^{(\infty)}. \tag{3.8}$$

Here is the result for the statistics obtained by the random truncation.

Theorem 3.8. (1) *(Quenched) For almost every ω, the push-forward of $p^{(n)}$ by the mapping*

$$U \mapsto n^{-1/2}\left(\mathcal{T}^{(n)}(\omega, U) - \frac{S^{(n)}(\omega) \otimes S'^{(n)}(\omega)}{n}\right)$$

converges weakly to $W^{(\infty)}$.

(2) *(Annealed) Under the joint probability measure $d\omega \otimes p^{(n)}(dU)$*

$$n^{-1/2}\left(\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)}\right) \xrightarrow{\text{law}} B_{00}\,. \tag{3.9}$$

4. Proofs by subordination

We present here proofs of Theorems 3.2, 3.5 and 3.8 whose key point is a representation by subordination.

4.1. **Preliminaries.**

Proposition 4.1. *Assume that U is a random unitary matrix such that the matrix whose generic entry is $|U_{ij}|^2$ has a distribution invariant by multiplication (right or left) by permutation matrix. Let $\widehat{T}^{(n)}$ be defined by*

$$\widehat{T}^{(n)}_{s,t}(\omega, U) = T^{(n)}_{n^{-1}S^{(n)}_s(\omega), n^{-1}S'^{(n)}_t(\omega)}(U)\,.$$

Then for every ω the push-forward of $\pi^{(n)}(dU)$ by the mapping $U \mapsto \mathcal{T}^{(n)}(\omega, U)$ is the same as the push-forward of $\pi^{(n)}(dU)$ by the mapping $U \mapsto \widehat{T}^{(n)}(\omega, U)$. As a result the law of $\mathcal{T}^{(n)}$ and $\widehat{T}^{(n)}$ have the same distribution under $d\omega \otimes \pi^{(n)}(dU)$.

Proof. Let $R = (R_1, \dots, R_n)$ and $C = (C_1, \dots, C_n)$ be two independent samples of uniform variables on $[0,1]$. The corresponding reordered samples are $\widetilde{R} = (R_{(1)}, \dots, R_{(n)})$ and $\widetilde{C} = (C_{(1)}, \dots, C_{(n)})$, and the associated random permutations are σ and τ, are defined by

$$R_{(i)} = R_{\sigma^{-1}(i)}\ ,\ C_{(j)} = C_{\tau^{-1}(j)},\ i,j = 1,\dots,n\,.$$

Moreover σ and $\widetilde{R}$ (resp. τ and $\widetilde{C}$) are independent. With these notations, we have

$$\begin{aligned}\mathcal{T}^{(n)}_{s,t} &= \sum_{i,j=1}^{n} |U_{ij}|^2 1_{R_i\le s} 1_{C_j \le t} = \sum_{i,j=1}^{n} |U_{\sigma^{-1}(i)\tau^{-1}(j)}|^2 1_{R_{(i)}\le s} 1_{C_{(j)}\le t} \\ &= \sum_{i\le S^{(n)}_s, j\le {S'}^{(n)}_t} |U_{\sigma^{-1}(i)\tau^{-1}(j)}|^2 = T^{(n)}_{n^{-1}S^{(n)}_s, n^{-1}{S'}^{(n)}_t}(\sigma U \tau^{-1})\,,\end{aligned}$$

where we have identified the permutations σ and τ with their matrices. Let F be some test function from $D([0,1]^2)$ to $\mathbb{R}$. We have

$$\mathbb{E}[F(\mathcal{T}^{(n)}(\omega, U))|\omega] = \mathbb{E}[F(T^{(n)}_{n^{-1}S^{(n)}, n^{-1}S'^{(n)}}(\sigma U\tau^{-1}))|\omega]\,.$$

Since the distribution of $(|U_{ij}|^2)^n_{i,j=1}$ is invariant by permutation we get

$$\mathbb{E}[F(\mathcal{T}^{(n)}(\omega, U)|\omega] = \mathbb{E}[F(T^{(n)}_{n^{-1}S^{(n)}, n^{-1}S'^{(n)}}(U))|\omega]$$

or, in other words

$$\mathbb{E}[F(\mathcal{T}^{(n)}(\omega, U)|\omega] = \mathbb{E}[F(\widehat{T}^{(n)}(\omega, U)|\omega]\,,$$

which ends the proof. $\square$

Now, the key point to manage the subordination of processes is the following proposition.

Proposition 4.2. *Let d be 1 or 2 and let $A^{(n)}$ be a sequence of processes with values in $D([0,1]^d)$ such that $A^{(n)} \xrightarrow{\text{law}} A$. Let $S^{(n)}$ and ${S'}^{(n)}$ be two independent processes defined as in (2.4) and independent of $A^{(n)}$.*

- *If $d = 1$, set $\mathcal{A}^{(n)} = \left(A^{(n)}\left(n^{-1}S^{(n)}_s\right)\right)\ ,\ s \in [0,1]\Big)$. Then*

$$\left(\mathcal{A}^{(n)}, \widetilde{S}^{(n)}\right) \xrightarrow{\text{law}} (A, B_0);$$

- *If $d = 2$, set $\mathcal{A}^{(n)} = \left(A^{(n)}\left(n^{-1}S^{(n)}_s, n^{-1}{S'}^{(n)}_t\right)\ ,\ s,t \in [0,1]\right)$. Then*

$$\left(\mathcal{A}^{(n)}, \widetilde{S}^{(n)}, \widetilde{S'}^{(n)}\right) \xrightarrow{\text{law}} (A, B_0, B_0'),$$

where A, B_0, B_0' are independent and B_0 and B_0' are two independent (one-parameter) Brownian bridges.

Notice that the marginal convergence of $\widetilde{S}^{(n)}$ (or of $(\widetilde{S}^{(n)}, \widetilde{S'}^{(n)})$) is nothing but Donsker's theorem.

Proof. Let us restrict us to the case $d = 1$ for simplicity. We follow the lines of proof of Theorem 1.6 of Wu [24]. According to the Skorokhod representation theorem, we can build a probability space and stochastic processes $\mathbf{A}^{(n)}$, $\mathbf{A}$, $\widetilde{\mathbf{S}}^{(n)}$, $\mathbf{B}_0$ on it such that

- all processes are $D([0,1])$ valued
- $\mathbf{A}^{(n)}$ and $\widetilde{\mathbf{S}}^{(n)}$ are independent and $\mathbf{A}^{(n)} \overset{\text{law}}{=} A^{(n)}$, $\widetilde{\mathbf{S}}^{(n)} \overset{\text{law}}{=} \widetilde{S}^{(n)}$

- $\mathbf{A}$ and $\mathbf{B}_0$ are independent and $\mathbf{A} \overset{\text{law}}{=} A$, $\mathbf{B}_0 \overset{\text{law}}{=} B_0$
- $\mathbf{A}^{(n)}$ and $\widetilde{\mathbf{S}}^{(n)}$ converge a.s. to $\mathbf{A}$ and $\mathbf{B}_0$, respectively.

Set $\mathbf{S}^{(n)} = n^{1/2}\widetilde{\mathbf{S}}^{(n)} + nI$. The convergence a.s. of $\widetilde{\mathbf{S}}^{(n)}$ entails the convergence a.s. of $n^{-1}\mathbf{S}^{(n)}$ to I as a $D([0,1])$-valued non-decreasing process. Here the limiting subordinator is continuous. We are exactly in the conditions of Theorem 1.2 of [24], which allows to say

$$a.s. - \lim_n \mathbf{A}_n \circ \mathbf{S}^{(n)} = \mathbf{A}$$

and of course

$$a.s. - \lim_n (\mathbf{A}_n \circ \mathbf{S}^{(n)}, \widetilde{\mathbf{S}}^{(n)}) = (\mathbf{A}, \mathbf{B}_0)\,.$$

Now, we conclude, going down to the convergence in distribution,

$$(\mathcal{A}_n, \widetilde{S}^{(n)}) \xrightarrow{\text{law}} (A, B_0)\,,$$

where A and B_0 are independent. $\square$

4.2. **Proof of Theorem 3.2.** From Proposition 4.1 (stated for one-parameter processes), we have the equality in law (conditionally on ω)

$$\{\mathcal{B}_s^{(n)}(\omega, .)\,, s \in [0,1]\} \overset{\text{law}}{=} \{B^{(n)}_{n^{-1}S_s^{(n)}(\omega)}(.), s \in [0,1]\} \tag{4.1}$$

and then we decompose

$$n^{1/2}\left(B^{(n)}_{n^{-1}S_s^{(n)}(\omega)} - s\right) = n^{1/2}\left(B^{(n)}_{n^{-1}S_s^{(n)}(\omega)} - n^{-1}S_s^{(n)}\right) + \widetilde{S}_s^{(n)}\,. \tag{4.2}$$

If we set $A^{(n)}(s) = n^{1/2}\left(B_s^{(n)} - n^{-1}\lfloor ns\rfloor\right)$, Lemma 3.1 above says that we are exactly in the assumptions of Proposition 4.2. Both processes of the RHS of (4.2) converge in distribution towards two independent processes, distributed as $\sqrt{\beta'^{-1}}B_0$ and B_0 respectively, hence the sum converges in distribution to $\sqrt{1+\beta'^{-1}}B_0$. $\square$

4.3. **Proofs of Theorems 3.5 and 3.8.** From Proposition 4.1, we reduce the problems to the study of $\widehat{T}^{(n)}$. Let us first remark that

$$\mathbb{E}[\mathcal{T}^{(n)}|\omega] = \mathbb{E}[\widehat{T}^{(n)}|\omega] = n^{-1}S^{(n)}(\omega) \otimes S'^{(n)}(\omega)\,. \tag{4.3}$$

If we set

$$\widehat{W}^{(n)}(\omega, U) = \widehat{T}^{(n)}(\omega, U) - \mathbb{E}[\widehat{T}^{(n)}|\omega] \tag{4.4}$$

we have the decomposition:

$$\widehat{T}^{(n)} - \mathbb{E}\widehat{T}^{(n)} = \widehat{W}^{(n)} + \widetilde{S}^{(n)} \otimes \widetilde{S'}^{(n)} + n^{1/2}\left(I \otimes \widetilde{S'}^{(n)} + \widetilde{S}^{(n)} \otimes I\right)\,. \tag{4.5}$$

4.3.1. *Proof of Theorem 3.5.* For the quenched fluctuations of $\widehat{W}^{(n)}$, we are exactly in the assumptions of Proposition 4.2, with $A^{(n)} = \widehat{W}^{(n)}$ and $A = \sqrt{\beta'^{-1}} W^{(\infty)}$, thanks to Theorem 3.4. This implies in particular that *(1)* holds.

For *(2)*, from Proposition 4.2, we see also that

$$(\widehat{W}^{(n)}, \widetilde{S}^{(n)}, \widetilde{S'}^{(n)}) \xrightarrow{\text{law}} (\sqrt{\beta'^{-1}} W^{(\infty)}, B_0, B_0') \tag{4.6}$$

where the three processes are independent. This implies

$$\widehat{W}^{(n)} + \widetilde{S}^{(n)} \otimes \widetilde{S'}^{(n)} \xrightarrow{\text{law}} \sqrt{\beta'^{-1}} W^{(\infty)} + B_0 \otimes B_0'$$

and consequently

$$n^{-1/2} \left(\widehat{W}^{(n)} + \widetilde{S}^{(n)} \otimes \widetilde{S'}^{(n)} \right) \to 0$$

in probability. Looking at the decomposition (4.5) and using again the convergence

$$I \otimes \widetilde{S'}^{(n)} + \widetilde{S}^{(n)} \otimes I \xrightarrow{\text{law}} \mathcal{W}^{(\infty)}$$

we conclude that

$$n^{-1/2} \left(\widehat{T}^{(n)} - \mathbb{E}\widehat{T}^{(n)} \right) \xrightarrow{\text{law}} \mathcal{W}^{(\infty)}$$

which is equivalent to the statement of *(2)*. □

4.3.2. *Proof of Theorem 3.8.* We are now in the assumptions of Proposition 4.2, with $A^{(n)} = n^{-1/2}(T^{(n)} - \mathbb{E}(T^{(n)}))$ and $A = W^{(\infty)}$ thanks to Theorem 3.7. This implies in particular that *(1)* holds.

For *(2)*, from Proposition 4.2, we see also that

$$\left(n^{-1/2} \widehat{W}^{(n)}, \widetilde{S}^{(n)}, \widetilde{S'}^{(n)} \right) \xrightarrow{\text{law}} (W^{(\infty)}, B_0, B_0') \tag{4.7}$$

where the three processes are independent. This implies

$$n^{-1/2} \left(\widehat{T}^{(n)} - \mathbb{E}\widehat{T}^{(n)} \right) \xrightarrow{\text{law}} W^{(\infty)} + \mathcal{W}^{(\infty)},$$

where the two processes in the RHS are independent. The equality in law

$$B_{0,0} \overset{\text{law}}{=} W^{\infty} + \mathcal{W}^{\infty}$$

was quoted in [13] section 2. □

Remark 4.3. *In the way leading from Theorem 3.4 to Theorem 3.5, we can see that the Haar distribution of random unitary matrices is not involved in the proofs, except via the invariance in law by permutation of rows and columns (which is the core of Proposition 4.1). In a recent work, Benaych-Georges* [3] *proved that, under some conditions, the unitary matrix of eigenvectors of a Wigner matrix induces the same behavior for the asymptotics of $T^{(n)}$ as under the Haar distribution. In the same vein, Bouferroum* [7] *proved a similar statement for the unitary matrix of eigenvectors of a sample covariance matrix. We could ask if it is possible to take benefit of these results to give an extension of our theorem to more general random unitary* *matrices. Actually, these authors proved that if the eigenvalues are ordered increasingly and if we call* $U_<$ *the matrix of corresponding eigenvectors, under some assumptions, the process* $T^{(n)}(U_<) - \mathbb{E}T^{(n)}(U_<)$ *converges in law to* $\sqrt{\beta'^{-1}}W^{(\infty)}$ *as in Theorem 3.4. To be able to apply Proposition 4.1, we would have to check the invariance of the law of the matrix* $(|(U_<)_{ij}|^2)$ *under multiplication (left or right) by a permutation matrix. A short look reveals that if* σ *is a permutation, then in both models* $\sigma M^{(n)}\sigma^*$ *and* $M^{(n)}$ *share the same eigenvalues, and* $\sigma U_<$ *is the matrix of eigenvalues of* $\sigma M^{(n)}\sigma^*$*. But they may not have the same distribution if they are complex. Even if we restrict to real Wigner matrices, we have indeed* $\sigma U_< \overset{\text{law}}{=} U_<$ *but the other type of equality in law (right permutation) is in general not true. For further remarks on the type of unitary matrices which could give the same convergence, see Section 5.*

5. About direct proofs of the main results and two conjectures

First, let us remark that in the Haar and permutation models, the key tool of the above approach to random truncation was the subordination machine. It assumes that we know the previous results on the deterministic truncation. Going back to the proof of this latter result in the Haar case [14], we see that estimates of moments of all degrees of monomials in entries of the unitary matrix are needed. We can ask if a direct method to tackle the random truncation demands so high moments estimates. This fact, among others, legitimates an interest for direct proofs, starting from the representation (2.3) in the Haar and permutation models and from the representation (2.2) for the one-dimensional process.

A second striking fact in the study of the two-parameter process is that in Theorems 3.3 and 3.5(2), the limiting processes are the same. In other words, the behavior of the sequence of DFT matrices is the same as the mean behavior of sequence of Haar matrices. If we define $\mathbb{U}(\infty) := \times_{n=1}^{\infty}\mathbb{U}(n)$, we can then ask how large is the set

$$\mathcal{E} := \{u = (U^{(n)}, n \in \mathbb{N}) \in \mathbb{U}(\infty) \mid n^{-1/2}(\mathcal{T}^{(n)}(., U^{(n)}) - nI \otimes I) \xrightarrow{\text{law}} \mathcal{W}^{(\infty)}\}\,.$$

Actually, it is equivalent to consider ω as the random object and the collection of $\mathbb{U}(n), n \in \mathbb{N}$ as the space of environments. There are several choices to equip $\mathbb{U}(\infty)$ with a probability measure such that its marginal on $\mathbb{U}(n)$ is the Haar measure $\pi^{(n)}$. In [8], the authors introduced the notion of virtual isometry. They consider a family of projections $\wp_{m,n}$ from $\mathbb{U}^{(n)}$ to $\mathbb{U}^{(m)}$ for $m < n$ and define the subset $\mathcal{U}(\infty)$ of $\mathbb{U}(\infty)$ of u such that $\wp_{m,n}(U^{(n)}) = U^{(m)}$ for every m, n with $m < n$. They conclude that there exists a unique probability measure π on $\mathcal{U}(\infty)$ (equipped with the cylindrical σ-algebra) whose n-th marginal is $\pi^{(n)}$ for every n. Their construction is also compatible with the framework of permutations (replace $\pi^{(n)}$ by $p^{(n)}$ and π by p). It could be noticed that the DFT sequence belongs to $\mathbb{U}(\infty) \setminus \mathcal{U}(\infty)$. Besides, in [20], Jiang "inspired by a common statistical procedure for simulating a sequence of Haar distributed matrices in statistical programs" assume that $(U^{(n)}, n \in \mathbb{N})$ is an independent sequence. The same remarks hold for $\mathbb{O}(\infty) := \times_{n=1}^{\infty} \mathbb{O}(n)$.

In the following subsections, we will give alternate proofs of Theorems 3.2 and 3.5 (annealed). We propose also two conjectures about weak convergences conditionally upon u, for π - almost every $u \in \mathbb{U}(\infty)$ (resp. $\mathbb{O}(\infty)$), where π is any probability measure whose marginals are Haar measures on $\mathbb{U}(n)$ (resp. $\mathbb{O}(n)$). At last we treat the permutation process.

5.1. **The one-parameter process.**

5.1.1. *Alternate proof of Theorem 3.2 (annealed).* The process

$$\mathcal{G}^{(n)} := \left\{ n^{1/2} \sum_1^n |U_{i1}|^2 \left(1_{R_i \le s} - s\right) \ , \ s \in [0,1] \right\}$$

is an example of a so-called weighted empirical process. We could then apply Theorem 1.1 of Koul and Ossiander [21, p. 544], and use the representation of $(|U_{11}|^2, \dots, |U_{i1}|^2, \dots, |U_{in}|^2)$ by means of gamma variables to check conditions therein. But we prefer to give (the sketch of) a proof that is more self-contained and closer to what will happen in the two-dimensional case.

A possible method for the finite dimensional convergence of $\mathcal{G}^{(n)}$ is to use the Lindeberg strategy of replacement by Gaussian variables. It says that if $G_1, \dots, G_n$ are independent Brownian bridges, then $\{\mathcal{G}^{(n)}(s) \ , \ s \in [0,1]\}$ and $\{n^{1/2} \sum_1^n |U_{i1}|^2 G_i(s) \ , \ s \in [0,1]\}$ have the same limits if

$$\lim_n \mathbb{E}\left[\sum_1^n \left(n^{1/2}|U_{i1}|^2\right)^3\right] = 0 \tag{5.1}$$

which holds true since $\mathbb{E}|U_{i1}|^6 = O(n^{-3})$. Then it remains to see that

$$\left\{ n^{1/2} \sum_1^n |U_{i1}|^2 G_i(s) \ , \ s \in [0,1] \right\} \overset{\text{law}}{=} \left\{ \left(n \sum_1^n |U_{i1}|^4 \right)^{1/2} G_1(s) \ , \ s \in [0,1] \right\}$$

and to prove

$$\lim_n n \sum_1^n |U_{i1}|^4 = 1 + \beta'^{-1} \tag{5.2}$$

in probability. This latter task may be performed using moments of order one and two of the above expression. We skip the details.

To prove tightness, we revisit criterion (14.9) of Billingsley [6]. For $r < s < t$, we have

$$\mathbb{E}\left[\left(\mathcal{G}^{(n)}(s)-\mathcal{G}^{(n)}(r)\right)^2\left(\mathcal{G}^{(n)}(t)-\mathcal{G}^{(n)}(s)\right)^2\right] = O((s-r)(t-s)) \times\mathbb{E}\left(\sum_{i\neq j}|U_{i1}|^2|U_{j1}|^2+\sum_i |U_{i1}|^4\right) \quad (5.3)$$

Since $\sum_1^n |U_{i1}|^2 = 1$ and $\lim_n \mathbb{E}\left(n\sum_1^n |U_{i1}|^4\right) = 1+\beta'^{-1}$,we have

$$\sup_n \mathbb{E}\left(\sum_{i\neq j}|U_{i1}|^2|U_{j1}|^2+\sum_i |U_{i1}|^4\right) < \infty \quad (5.4)$$

and the proof is ended.

5.1.2. *Conjecture 1.* Inspecting the above proof, we see that if all the convergences and bounds for statistics built from U (i.e. (5.1), (5.2) and (5.4)) were almost sure, we could claim a quenched convergence in distribution. Of course, we probably need higher moments calculus.

Conjecture 1. For π - almost every u, the push-forward of $d\omega$ by the mapping

$$\omega \mapsto \sqrt{n}\left(\mathcal{B}^{(n)}(\omega, U) - I\right)$$

converges weakly to the distribution of $\sqrt{1+\beta'^{-1}}B_0$.

5.2. **The Haar process.**

5.2.1. *Alternate proof of Theorem 3.5 (annealed).* For a complete proof in this flavor, see [15]. We start from the following decomposition, analogous to (4.5):

$$\mathcal{T}^{(n)} - \mathbb{E}\mathcal{T}^{(n)} = \mathcal{V}^{(n)} + \widetilde{S}^{(n)}\otimes\widetilde{S'}^{(n)} + n^{1/2}\left(I\otimes\widetilde{S'}^{(n)} + \widetilde{S}^{(n)}\otimes I\right) \quad (5.5)$$

where $\mathcal{V}^{(n)}$ is defined in (3.7), or explicitly by

$$\mathcal{V}^{(n)}_{s,t}(\omega, U) = \sum_{ij}\left(|U_{ij}|^2 - n^{-1}\right)[\mathbf{1}_{R_i\leq s} - s][\mathbf{1}_{C_j\leq t} - t] \quad (5.6)$$

(compare with $\widehat{W}^{(n)}$). The scheme consists in proving

1. the convergence of $\mathcal{V}^{(n)}$ to $\sqrt{\beta'^{-1}}W^{[\infty)}$ in the sense of finite dimensional distributions;
2. the tightness of the sequence $n^{-1/2}\mathcal{V}^{(n)}$ in $D([0,1]^2)$.

It seems to be similar to the one above in the one-dimensional case, nevertheless there are two differences. First, we do not study $\mathcal{T}^{(n)}$, but $\mathcal{V}^{(n)}$ which is its "involved" part. Second, we did not succeed to prove directly the tightness of $\mathcal{V}^{(n)}$ but only that of $n^{-1/2}\mathcal{V}^{(n)}$. Actually we know that a stronger result holds true, since as a consequence of Theorem 3.5 (quenched), $\mathcal{V}^{(n)} \xrightarrow{\text{law}} \sqrt{\beta'^{-1}} W^{(\infty)}$, but using Theorem 3.5 here defeats the point.

For the finite dimensional convergence we use again the Lindeberg strategy replacing first the processes $1_{R_i \le s} - s$ by Brownian bridges $\beta_i(s)$ and afterwards replacing the processes $1_{C_j \le t} - t$ by Brownian bridges $\widetilde{\beta}_j(t)$. The original process and the new one have the same limit as soon as

$$\lim_n \mathbb{E} \sum_{i,j=1}^n |U_{ij}|^6 = 0 \tag{5.7}$$

(which is true, since again $\mathbb{E}|U_{ij}|^6 = O(n^{-3})$). To simplify, let us explain what happens for the one-dimensional marginal after the above replacement.

Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$, two independent standard Gaussian vectors in $\mathbb{R}^n$. We thus study the bilinear non symmetric form

$$Q_n := \sum_{i,j=1}^n X_i (|U_{ij}|^2 - 1/n) Y_j$$

built from the non symmetric matrix $V = \left(|U_{ij}|^2 - 1/n\right)_{i,j \le n}$. The characteristic function of Q_n is computed by conditioning upon X and U; taking into account the Gaussian distribution of Y we are lead to study the quadratic form

$$\widehat{Q}_n := \sum_{i,j=1}^n X_i H_{ij} X_j \ , \ H_{ij} = (VV^*)_{ij} \ , \ i,j = 1, \dots, n$$

and we have to prove that

$$\lim_n \widehat{Q}_n = \beta'^{-1} , \tag{5.8}$$

in probability. It can be checked with straightforward calculations of moments of order 1 and 2 of $\widehat{Q}_n$, demanding moments of order 8 of the entries of U to be computed.

To prove the tightness of $n^{-1/2}\mathcal{V}^{(n)}$ we can use a criterion of Davydov and Zitikis [12] (notice that several criteria for tightness known in the literature, such as Bickel-Wichura [5] or Ivanoff [19], failed in this model). A sufficient condition is:

$$\mathbb{E} \left(\mathcal{V}_{s,t}^{(n)} \right)^6 \le C (\max(s,t))^3 \tag{5.9}$$

as soon as $\max(s,t) \ge n^{-1}$. With a careful look at dependencies, we reach:

$$\mathbb{E}^U \left(\mathcal{V}_{s,t}^{(n)} \right)^6 = \sum_{i_k, j_k, k=1,\dots 6} \left(\prod_{k=1}^6 V_{i_k j_k}\right) \mathbb{E}\left(\prod_{k=1}^6 B_{i_k}\right) \mathbb{E}\left(\prod_{k=1}^6 B'_{j_k}\right). \tag{5.10}$$

Since the B_i and B'_j are independent and centered, in the RHS of the above equation, the non-zero terms in the sum are obtained when the i_k (resp. the j_k) are equal at least 2 by 2. Using the following properties:

- $|\,\mathbb{E}((B_i)^k)| \leq \mathbb{E}(B_i)^2 \leq s$ for $2 \leq k \leq 6$
- $|\,\mathbb{E}((B'_j)^k)| \leq \mathbb{E}(B'_j)^2 \leq t$ for $2 \leq k \leq 6\,,$

it was checked in [15] that (5.9) holds true.

5.2.2. *Conjecture 2.* Inspecting the above proof, we see that if all the convergences and bounds for statistics built from U (i.e. (5.7), (5.8) and (5.9)) were almost sure, we could claim a quenched convergence in distribution. This would require sharp analysis of homogeneous polynomials in the entries of V hence of U.

Conjecture 2. For π - almost every u, the push-forward of $d\omega$ by the map

$$\omega \mapsto n^{-1/2}\left(\mathcal{T}^{(n)}(\omega, U^{(n)}) - nI \otimes I\right)$$

converges weakly to the distribution of $\mathcal{W}^{(\infty)}$. In other words $\pi(\mathcal{E}) = 1$.

5.3. **The permutation process.** For the permutation process we have a complete picture, i.e. a convergence conditionally upon u, hence a direct proof of Theorem 3.8 (annealed).

Theorem 5.1. *For every $u \in \otimes_{n=1}^{\infty}\mathcal{S}_n$, the push-forward of $d\omega$ by the mapping*

$$\omega \mapsto n^{-1/2}\left(\mathcal{T}^{(n)}(\omega, U^{(n)}) - nI \otimes I\right)$$

converges weakly to the distribution of B_{00}.

Proof. If $\sigma^{(n)}$ is the permutation associated with $U^{(n)}$, we have

$$\mathcal{T}_{st}^{(n)} = \sum_{i=1}^{n} \mathbf{1}_{R_i \leq s}\mathbf{1}_{C_{\sigma^{(n)}(i)} \leq t}\,.$$

If we fix u, we fix $\sigma^{(n)}$ for every n. It is clear that the sequence $C_{\sigma^{(n)}(i)}, 1 \leq i \leq n$ has the same distribution as $C_i, 1 \leq i \leq n$. We have then (conditionally)

$$n^{-1/2}(\mathcal{T}^{(n)}(., U^{(n)}) - nI \otimes I) \stackrel{\text{law}}{=} \mathcal{X}_n$$

where

$$\mathcal{X}_n := \left(n^{-1/2}\left(\sum_{i=1}^{n} \mathbf{1}_{R_i \leq s}\mathbf{1}_{C_i \leq t} - nst\right)\ ,\ s, t \in [0,1]\right)$$

is a classical two-parameter empirical process. In [22] it is proved that this process converges in distribution to B_{00}. If $\mathcal{F}$ is any bounded continuous function of $D([0,1)^2)$ in $\mathbb{R}$ we may write, for every u,

$$\mathbb{E}\left[\mathcal{F}\left(n^{-1/2}(\mathcal{T}^{(n)}(., U^{(n)}) - nI \otimes I)\right)\ \Big|\ u\right] = \mathbb{E}\mathcal{F}(\mathcal{X}_n)) \to_{n\to\infty} \mathbb{E}\mathcal{F}(B_{00})$$

which concludes the proof. $\square$

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