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On the critical value function in the divide and color model

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Abstract. The divide and color model on a graph G arises by first deleting each edge of G with probability $1-p$ independently of each other, then coloring the resulting connected components (*i.e.*, every vertex in the component) black or white with respective probabilities r and $1-r$, independently for different components. Viewing it as a (dependent) site percolation model, one can define the critical point $r_c^G(p)$.

In this paper, we mainly study the continuity properties of the function r_c^G, which is an instance of the question of locality for percolation. Our main result is the fact that in the case $G=\mathbb{Z}^2$, r_c^G is continuous on the interval $[0,1/2)$; we also prove continuity at $p=0$ for the more general class of graphs with bounded degree. We then investigate the sharpness of the bounded degree condition and the monotonicity of $r_c^G(p)$ as a function of p.

1. Introduction

The divide and color (DaC) model is a natural dependent site percolation model introduced by Häggström (2001). It has been studied directly in Häggström (2001); Garet (2001); Bálint et al. (2009), and as a member of a more general family of models in Kahn and Weininger (2007); Bálint et al. (2009); Bálint (2010); Graham and Grimmett (2011). This model is defined on a multigraph $G=(\mathcal{V},\mathcal{E})$, where $\mathcal{E}$ is a multiset (*i.e.*, it may contain an element more than once), thus allowing parallel edges between pairs of vertices. For simplicity, we will imprecisely call G a *graph*

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and $\mathcal{E}$ the *edge set*, even if G contains self-loops or multiple edges. The DaC model with parameters $p, r \in [0,1]$, on a general (finite or infinite) graph G with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, is defined by the following two-step procedure:

- First step: Bernoulli bond percolation. We independently declare each edge in $\mathcal{E}$ to be open with probability p, and closed with probability $1-p$. We can identify a bond percolation configuration with an element $\eta \in \{0,1\}^{\mathcal{E}}$: for each $e \in \mathcal{E}$, we define $\eta(e) = 1$ if e is open, and $\eta(e) = 0$ if e is closed.
- Second step: Bernoulli site percolation on the resulting cluster set. Given $\eta \in \{0,1\}^{\mathcal{E}}$, we call *p-clusters* or *bond clusters* the connected components in the graph with vertex set $\mathcal{V}$ and edge set $\{e \in \mathcal{E} : \eta(e) = 1\}$. The set of p-clusters of η gives a partition of $\mathcal{V}$. For each p-cluster $\mathcal{C}$, we assign the same color to all the vertices in $\mathcal{C}$. The chosen color is black with probability r and white with probability $1-r$, and this choice is independent for different p-clusters.

These two steps yield a site percolation configuration $\xi \in \{0,1\}^{\mathcal{V}}$ by defining, for each $v \in \mathcal{V}$, $\xi(v) = 1$ if v is black, and $\xi(v) = 0$ if v is white. The connected components (via the edge set $\mathcal{E}$) in ξ of the same color are called (black or white) *r-clusters*. The resulting measure on $\{0,1\}^{\mathcal{V}}$ is denoted by $\mu_{p,r}^{G}$.

Let $E_\infty^b \subset \{0,1\}^{\mathcal{V}}$ denote the event that there exists an infinite black r-cluster. By standard arguments (see Proposition 2.5 in Häggström (2001)), for each $p \in [0,1]$, there exists a *critical coloring value* $r_c^G(p) \in [0,1]$ such that

$$\mu_{p,r}^{G}(E_\infty^b) \begin{cases} = 0 & \text{if } r < r_c^G(p), \\ > 0 & \text{if } r > r_c^G(p). \end{cases}$$

The *critical edge parameter* $p_c^G \in [0,1]$ is defined as follows: the probability that there exists an infinite bond cluster is 0 for all $p < p_c^G$, and positive for all $p > p_c^G$. The latter probability is in fact 1 for all $p > p_c^G$, whence $r_c^G(p) = 0$ for all such p. Kolmogorov's $0-1$ law shows that in the case when all the bond clusters are finite, $\mu_{p,r}^{G}(E_\infty^b) \in \{0,1\}$; nevertheless it is possible that $\mu_{p,r}^{G}(E_\infty^b) \in (0,1)$ for some $r > r_c^G(p)$ (*e.g.* on the square lattice, as soon as $p > p_c = 1/2$, one has $\mu_{p,r}^{G}(E_\infty^b) = r$).

Statement of the results. Our main goal in this paper is to understand how the critical coloring parameter r_c^G depends on the edge parameter p. Since the addition or removal of self-loops obviously does not affect the value of $r_c^G(p)$, we will assume that all the graphs G that we consider are without self-loops. On the other hand, G is allowed to contain multiple edges.

Our first result, based on a stochastic domination argument, gives bounds on $r_c^G(p)$ in terms of $r_c^G(0)$, which is simply the critical value for Bernoulli site percolation on G. By the *degree* of a vertex v, we mean the number of edges incident on v (counted with multiplicity).

Proposition 1.1. *For any graph G with maximal degree Δ, for all $p \in [0,1)$,*

$$1 - \frac{1 - r_c^G(0)}{(1-p)^\Delta} \le r_c^G(p) \le \frac{r_c^G(0)}{(1-p)^\Delta}.$$

As a direct consequence, we get continuity at $p = 0$ of the critical value function:

Proposition 1.2. *For any graph G with bounded degree, $r_c^G(p)$ is continuous in p at 0.*

One could think of an alternative approach to the question, as follows: the DaC model can be seen as Bernoulli site percolation of the random graph $G_p = (V_p, E_p)$ where V_p is the set of bond clusters and two bond clusters are connected by a bond of E_p if and only if they are adjacent in the original graph. The study of how $r_c^G(p)$ depends on p is then a particular case of a more general question known as the *locality problem*: is it true in general that the critical points of site percolation on a graph and a small perturbation of it are always close? Here, for small p, the graphs G and G_p are somehow very similar, and their critical points are indeed close.

Dropping the bounded-degree assumption allows for the easy construction of graphs for which continuity does not hold at $p = 0$:

Proposition 1.3. *There exists a graph G with $p_c^G > 0$ such that r_c^G is discontinuous at 0.*

In general, when $p > 0$, the graph G_p does not have bounded degree, even if G does; this simple remark can be exploited to construct bounded degree graphs for which r_c^G has discontinuities below the critical point of bond percolation (though of course not at 0):

Theorem 1.4. *There exists a graph G of bounded degree satisfying $p_c^G > 1/2$ and such that $r_c^G(p)$ is discontinuous at $1/2$.*

Remark 1.5. The value $1/2$ in the statement above is not special: in fact, for every $p_0 \in (0, 1)$, it is possible to generalize our argument to construct a graph with a critical bond parameter above p_0 and for which the discontinuity of r_c occurs at p_0.

Our main results concerns the case $G = \mathbb{Z}^2$, for which the above does not occur:

Theorem 1.6. *The critical coloring value $r_c^{\mathbb{Z}^2}(p)$ is a continuous function of p on the whole interval $[0, 1/2)$.*

The other, perhaps more anecdotal question we investigate here is whether r_c^G is monotonic below p_c. This is the case on the triangular lattice (because it is constant equal to $1/2$), and appears to hold on $\mathbb{Z}^2$ in simulations (see the companion paper Bálint et al. (2013)).

In the general case, the question seems to be rather delicate. Intuitively the presence of open edges would seem to make percolation easier, leading to the intuition that the function $p \mapsto r_c(p)$ should be nonincreasing. Theorem 2.9 in Häggström (2001) gives a counterexample to this intuition. It is even possible to construct quasi-transitive graphs on which any monotonicity fails:

Proposition 1.7. *There exists a quasi-transitive graph G such that r_c^G is not monotone on the interval $[0, p_c^G)$.*

A brief outline of the paper is as follows. We set the notation and collect a few results from the literature in Section 2. In Section 3, we stochastically compare $\mu_{p,r}^G$ with Bernoulli site percolation (Theorem 3.1), and show how this result implies Proposition 1.1. We then turn to the proof of Theorem 1.6 in Section 4, based on a finite-size argument and the continuity of the probability of cylindrical events.

In Section 5, we determine the critical value function for a class of tree-like graphs, and in the following section we apply this to construct most of the examples of graphs we mentioned above.

2. Definitions and notation

We start by explicitly constructing the model, in a way which will be more technically convenient than the intuitive one given in the introduction.

Let G be a connected graph $(\mathcal{V}, \mathcal{E})$ where the set of vertices $\mathcal{V} = \{v_0, v_1, v_2, \ldots\}$ is countable. We define a total order "$<$" on $\mathcal{V}$ by saying that $v_i < v_j$ if and only if $i < j$. In this way, for any subset $V \subset \mathcal{V}$, we can uniquely define $\min(V) \in V$ as the minimal vertex in V with respect to the relation "$<$". For a set S, we denote $\{0,1\}^S$ by Ω_S. We call the elements of $\Omega_\mathcal{E}$ *bond configurations*, and the elements of $\Omega_\mathcal{V}$ *site configurations*. As defined in the Introduction, in a bond configuration η, an edge $e \in \mathcal{E}$ is called *open* if $\eta(e) = 1$, and *closed* otherwise; in a site configuration ξ, a vertex $v \in \mathcal{V}$ is called *black* if $\xi(e) = 1$, and *white* otherwise. Finally, for $\eta \in \Omega_\mathcal{E}$ and $v \in \mathcal{V}$, we define the *bond cluster* $\mathcal{C}_v(\eta)$ of v as the maximal connected induced subgraph containing v of the graph with vertex set $\mathcal{V}$ and edge set $\{e \in \mathcal{E} : \eta(e) = 1\}$, and denote the vertex set of $\mathcal{C}_v(\eta)$ by $C_v(\eta)$.

For $a \in [0,1]$ and a set S, we define ν_a^S as the probability measure on Ω_S that assigns to each $s \in S$ value 1 with probability a and 0 with probability $1-a$, independently for different elements of S. We define a function

$$\begin{array}{rccc} \Phi: & \Omega_\mathcal{E} \times \Omega_\mathcal{V} & \to & \Omega_\mathcal{E} \times \Omega_\mathcal{V}, \\ & (\eta, \kappa) & \mapsto & (\eta, \xi), \end{array}$$

where $\xi(v) = \kappa(\min(C_v(\eta)))$. For $p, r \in [0,1]$, we define $\mathbb{P}_{p,r}^G$ to be the image measure of $\nu_p^\mathcal{E} \otimes \nu_r^\mathcal{V}$ by the function Φ, and denote by $\mu_{p,r}^G$ the marginal of $\mathbb{P}_{p,r}^G$ on $\Omega_\mathcal{V}$. Note that this definition of $\mu_{p,r}^G$ is consistent with the one in the Introduction.

Finally, we give a few definitions and results that are necessary for the analysis of the DaC model on the square lattice, that is the graph with vertex set $\mathbb{Z}^2$ and edge set $\mathcal{E}^2 = \{\langle v, w\rangle : v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{Z}^2,\ |v_1 - w_1| + |v_2 - w_2| = 1\}$. The *matching graph* $\mathbb{Z}^2_*$ of the square lattice is the graph with vertex set $\mathbb{Z}^2$ and edge set $\mathcal{E}^2_* = \{\langle v, w\rangle : v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{Z}^2,\ \max(|v_1 - w_1|, |v_2 - w_2|) = 1\}$. In the same manner as in the Introduction, we define, for a color configuration $\xi \in \{0,1\}^{\mathbb{Z}^2}$, (black or white) $*$-*clusters* as connected components (via the edge set $\mathcal{E}^2_*$) in ξ of the same color. We denote by $\Theta^*(p,r)$ the $\mathbb{P}_{p,r}^{\mathbb{Z}^2}$-probability that the origin is contained in an infinite black $*$-cluster, and define

$$r_c^*(p) = \sup\{r : \Theta^*(p,r) = 0\}$$

for all $p \in [0,1]$ — note that this value may differ from $r_c^{\mathbb{Z}^2_*}(p)$. The main result in Bálint et al. (2009) is that for all $p \in [0, 1/2)$, the critical values $r_c^{\mathbb{Z}^2}(p)$ and $r_c^*(p)$ satisfy the duality relation

$$r_c^{\mathbb{Z}^2}(p) + r_c^*(p) = 1. \tag{2.1}$$

We will also use exponential decay result for subcritical Bernoulli bond percolation on $\mathbb{Z}^2$. Let $\mathbf{0}$ denote the origin in $\mathbb{Z}^2$, and for each $n \in \mathbb{N} = \{1, 2, \ldots\}$, let us define $S_n = \{v \in \mathbb{Z}^2 : dist(v, \mathbf{0}) = n\}$ (where *dist* denotes graph distance), and the event $M_n = \{\eta \in \Omega_{\mathcal{E}^2} : \text{there is a path of open edges in } \eta \text{ from } \mathbf{0} \text{ to } S_n\}$. Then we have the following result:

Theorem 2.1 (Kesten (1980))**.** *For $p < 1/2$, there exists $\psi(p) > 0$ such that for all $n \in \mathbb{N}$, we have that*

$$\nu_p^{\mathcal{E}^2}(M_n) < e^{-n\psi(p)}.$$

3. Stochastic domination and continuity at $p = 0$

In this section, we prove Proposition 1.1 via a stochastic comparison between the DaC measure and Bernoulli site percolation. Before stating the corresponding result, however, let us recall the concept of stochastic domination.

We define a natural partial order on $\Omega_{\mathcal{V}}$ by saying that $\xi' \geq \xi$ for $\xi, \xi' \in \Omega_{\mathcal{V}}$ if, for all $v \in \mathcal{V}$, $\xi'(v) \geq \xi(v)$. A random variable $f : \Omega_{\mathcal{V}} \to \mathbb{R}$ is called *increasing* if $\xi' \geq \xi$ implies that $f(\xi') \geq f(\xi)$, and an event $E \subset \Omega_{\mathcal{V}}$ is increasing if its indicator random variable is increasing. For probability measures μ, μ' on $\Omega_{\mathcal{V}}$, we say that μ' is *stochastically larger* than μ (or, equivalently, that μ is *stochastically smaller* than μ', denoted by $\mu \leq_{\mathrm{st}} \mu'$) if, for all bounded increasing random variables $f : \Omega_{\mathcal{V}} \to \mathbb{R}$, we have that

$$\int_{\Omega_{\mathcal{V}}} f(\xi)\ d\mu'(\xi) \geq \int_{\Omega_{\mathcal{V}}} f(\xi)\ d\mu(\xi).$$

By Strassen's theorem (1965), this is equivalent to the existence of an appropriate coupling of the measures μ' and μ; that is, the existence of a probability measure $\mathbb{Q}$ on $\Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}$ such that the marginals of $\mathbb{Q}$ on the first and second coordinates are μ' and μ respectively, and $\mathbb{Q}(\{(\xi', \xi) \in \Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}} : \xi' \geq \xi\}) = 1$.

Theorem 3.1. *For any graph $G = (\mathcal{V}, \mathcal{E})$ whose maximal degree is Δ, at arbitrary values of the parameters $p, r \in [0, 1]$,*

$$\nu^{\mathcal{V}}_{r(1-p)^{\Delta}} \leq_{st} \mu^G_{p,r} \leq_{st} \nu^{\mathcal{V}}_{1-(1-r)(1-p)^{\Delta}}.$$

Before turning to the proof, we show how Theorem 3.1 implies Proposition 1.1. It follows from Theorem 3.1 and the definition of stochastic domination that for the increasing event E^b_∞ (which was defined in the Introduction), we have $\mu^G_{p,r}(E^b_\infty) > 0$ whenever $r(1-p)^{\Delta} > r^G_c(0)$, which implies that $r^G_c(p) \leq r^G_c(0)/(1-p)^{\Delta}$. The derivation of the lower bound for $r^G_c(p)$ is analogous. □

Now we give the proof of Theorem 3.1, which bears some resemblance with the proof of Theorem 2.3 in Häggström (2001). Fix $G = (\mathcal{V}, \mathcal{E})$ with maximal degree Δ, and parameter values $p, r \in [0, 1]$. We will use the relation "$<$" and the minimum of a vertex set with respect to this relation as defined in Section 2. In what follows, we will define several random variables; we will denote the joint distribution of all these variables by $\mathbb{P}$.

First, we define a collection $(\eta^e_{x,y} : x, y \in \mathcal{V}, e = \langle x, y\rangle \in \mathcal{E})$ of i.i.d. Bernoulli(p) random variables (*i.e.*, they take value 1 with probability p, and 0 otherwise); one may imagine having each edge $e \in \mathcal{E}$ replaced by two directed edges, and the random variables represent which of these edges are open. We define also a set $(\kappa_x : x \in \mathcal{V})$ of Bernoulli(r) random variables. Given a realization of $(\eta^e_{x,y} : x, y \in \mathcal{V}, e = \langle x, y\rangle \in \mathcal{E})$ and $(\kappa_x : x \in \mathcal{V})$, we will define an $\Omega_{\mathcal{V}} \times \Omega_{\mathcal{E}}$-valued random configuration (η, ξ) with distribution $\mathbb{P}^G_{p,r}$, by the following algorithm.

(1) Let $v = \min\{x \in \mathcal{V}$: no ξ-value has been assigned yet to x by this algorithm$\}$. (Note that v and V, v_i, H_i ($i \in \mathbb{N}$), defined below, are running variables, *i.e.*, their values will be redefined in the course of the algorithm.)

(2) We explore the "directed open cluster" V of v iteratively, as follows. Define $v_0 = v$. Given $v_0, v_1, \ldots, v_i$ for some integer $i \geq 0$, set $\eta(e) = \eta^e_{v_i,w}$ for every edge $e = \langle v_i, w\rangle \in \mathcal{E}$ incident to v_i such that no η-value has been assigned yet to e by the algorithm, and write $H_{i+1} = \{w \in \mathcal{V} \setminus \{v_0, v_1, \ldots, v_i\} : w$ can be reached from any of $v_0, v_1, \ldots, v_i$ by using only those edges $e \in \mathcal{E}$

such that $\eta(e) = 1$ has been assigned to e by this algorithm}. If $H_{i+1} \neq \emptyset$, then we define $v_{i+1} = \min(H_{i+1})$, and continue exploring the directed open cluster of v; otherwise, we define $V = \{v_0, v_1, \ldots, v_i\}$, and move to step 3.

(3) Define $\xi(w) = \kappa_v$ for all $w \in V$, and return to step 1.

It is immediately clear that the above algorithm eventually assigns a ξ-value to each vertex. Note also that a vertex v can receive a ξ-value only after all edges incident to v have already been assigned an η-value, which shows that the algorithm eventually determines the full edge configuration as well. It is easy to convince oneself that (η, ξ) obtained this way indeed has the desired distribution.

Now, for each $v \in \mathcal{V}$, we define $Z(v) = 1$ if $\kappa_v = 1$ and $\eta^e_{w,v} = 0$ for all edges $e = \langle v, w \rangle \in \mathcal{E}$ incident on v (*i.e.*, all directed edges towards v are closed), and $Z(v) = 0$ otherwise. Note that every vertex with $Z(v) = 1$ has $\xi(v) = 1$ as well, whence the distribution of ξ (*i.e.*, $\mu^G_{p,r}$) stochastically dominates the distribution of Z (as witnessed by the coupling $\mathbb{P}$).

Notice that $Z(v)$ depends only on the states of the edges pointing to v and on the value of κ_v; in particular the distribution of Z is a product measure on $\Omega_{\mathcal{V}}$ with parameter $r(1-p)^{d(v)}$ at v, where $d(v) \leq \Delta$ is the degree of v, whence $\mu^G_{p,r}$ stochastically dominates the product measure on $\Omega_{\mathcal{V}}$ with parameter $r(1-p)^{\Delta}$, which gives the desired stochastic lower bound. The upper bound can be proved analogously; alternatively, it follows from the lower bound by exchanging the roles of black and white. □

4. Continuity of $r_c^{\mathbb{Z}^2}(p)$ on the interval $[0, 1/2)$

In this section, we will prove Theorem 1.6. Our first task is to prove a technical result valid on more general graphs stating that the probability of any event A whose occurrence depends on a finite set of ξ-variables is a continuous function of p for $p < p^G_c$. The proof relies on the fact that although the color of a vertex v may be influenced by edges arbitrarily far away, if $p < p^G_c$, the corresponding influence decreases to 0 in the limit as we move away from v. Therefore, the occurrence of the event A depends essentially on a finite number of η- and κ-variables, whence its probability can be approximated up to an arbitrarily small error by a polynomial in p and r.

Once we have proved Proposition 4.1 below, which is valid on general graphs, we will apply it on $\mathbb{Z}^2$ to certain "box-crossing events," and appeal to results in Bálint et al. (2009) to deduce the continuity of $r_c^{\mathbb{Z}^2}(p)$.

Proposition 4.1. *For every site percolation event $A \subset \{0,1\}^{\mathcal{V}}$ depending on the color of finitely many vertices, $\mu^G_{p,r}(A)$ is a continuous function of (p, r) on the set $[0, p^G_c) \times [0, 1]$.*

Proof. In this proof, when μ is a measure on a set S, X is a random variable with law μ and $F : S \longrightarrow \mathbb{R}$ is a bounded measurable function, we write abusively $\mu[F(X)]$ for the expectation of $F(X)$. We show a slightly more general result: for any $k \geq 1$, $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathcal{V}^k$ and $f : \{0,1\}^k \to \mathbb{R}$ bounded and measurable, $\mu^G_{p,r}[f(\xi(x_1), \ldots, \xi(x_k))]$ is continuous in (p, r) on the product $[0, p^G_c) \times [0, 1]$. Proposition 4.1 will follow by choosing an appropriate family $\{x_1, \ldots, x_k\}$ such that the states of the x_i suffices to determine whether A occurs, and take f to be the indicator function of A.

To show the previous affirmation, we condition on the vector

$$\boldsymbol{m}_{\boldsymbol{x}}(\eta) = (\min C_{x_1}(\eta), \ldots, \min C_{x_k}(\eta))$$

which takes values in the finite set

$$\boldsymbol{V} = \left\{(v_1, \ldots, v_k) \in \mathcal{V}^k \ : \ \forall i \ v_i \leq \max\{x_1, \ldots, x_k\}\right\},$$

and we use the definition of $\mathbb{P}^G_{p,r}$ as an image measure. By definition,

$$\begin{aligned}
&\mu^G_{p,r}\left[f(\xi(x_1), \ldots, \xi(x_k))\right] \\
&\quad = \sum_{\boldsymbol{v}\in\boldsymbol{V}} \mathbb{P}^G_{p,r}\left[f(\xi(x_1), \ldots, \xi(x_k))|\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\}\right] \mathbb{P}^G_{p,r}\left[\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\}\right] \\
&\quad = \sum_{\boldsymbol{v}\in\boldsymbol{V}} \nu^{\mathcal{E}}_p \otimes \nu^{\mathcal{V}}_r\left[f(\kappa(v_1), \ldots, \kappa(v_k))|\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\}\right] \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\}\right] \\
&\quad = \sum_{\boldsymbol{v}\in\boldsymbol{V}} \nu^{\mathcal{V}}_r\left[f(\kappa(v_1), \ldots, \kappa(v_k))\right] \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\}\right].
\end{aligned}$$

Note that $\nu^{\mathcal{V}}_r\left[f(\kappa(v_1), \ldots, \kappa(v_k))\right]$ is a polynomial in r, so to conclude the proof we only need to prove that for any fixed $\boldsymbol{x}$ and $\boldsymbol{v}$, $\nu^{\mathcal{E}}_p\left(\{\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v}\}\right)$ depends continuously on p on the interval $[0, p^G_c)$.

For $n \geq 1$, write $F_n = \{|C_{x_1}| \leq n, \ldots, |C_{x_k}| \leq n\}$. It is easy to verify that the event $\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\} \cap F_n$ depends on the state of finitely many edges. Hence, $\nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\} \cap F_n\right]$ is a polynomial function of p.

Fix $p_0 < p^G_c$. For all $p \leq p_0$,

$$\begin{aligned}
0 \leq \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v}\}\right] - \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}_{\boldsymbol{x}} = \boldsymbol{v}\} \cap F_n\right] \quad &\leq \quad \nu^{\mathcal{E}}_p\left[F^c_n\right] \\
&\leq \quad \nu^{\mathcal{E}}_{p_0}\left[F^c_n\right]
\end{aligned}$$

where $\lim_{n\to\infty} \nu^{\mathcal{E}}_{p_0}\left[F^c_n\right] = 0$, since $p_0 < p^G_c$. So, $\nu^{\mathcal{E}}_p\left[\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v}\right]$ is a uniform limit of polynomials on any interval $[0, p_0]$, $p_0 < p^G_c$, which implies the desired continuity. □

Remark 4.2. In the proof we can see that, for fixed $p < p^G_c$, $\mu^G_{p,r}(A)$ is a polynomial in r.

Remark 4.3. If G is a graph with uniqueness of the infinite bond cluster in the supercritical regime, then it is possible to verify that $\nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v}\}\right]$ is continuous in p on the whole interval $[0, 1]$. In this case, the continuity given by the Proposition 4.1 can be extended to the whole square $[0, 1]^2$.

Proof of Theorem 1.6. In order to simplify our notations, we write $\mathbb{P}_{p,r}, \nu_p$, $r_c(p)$, for $\mathbb{P}^{\mathbb{Z}^2}_{p,r}, \nu^{\mathcal{E}^2}_p$ and $r^{\mathbb{Z}^2}_c(p)$ respectively. Fix $p_0 \in (0, 1/2)$ and $\varepsilon > 0$ arbitrarily. We will show that there exists $\delta = \delta(p_0, \varepsilon) > 0$ such that for all $p \in (p_0 - \delta, p_0 + \delta)$,

$$r_c(p) \geq r_c(p_0) - \varepsilon, \tag{4.1}$$

and

$$r_c(p) \leq r_c(p_0) + \varepsilon. \tag{4.2}$$

Note that by equation (2.1), for all small enough choices of $\delta > 0$ (such that $0 \leq p_0 \pm \delta < 1/2$), (4.1) is equivalent to

$$r^*_c(p) \leq r^*_c(p_0) + \varepsilon. \tag{4.3}$$

Below we will show how to find $\delta_1 > 0$ such that we have (4.2) for all $p \in (p_0 - \delta_1, p_0 + \delta_1)$. One may then completely analogously find $\delta_2 > 0$ such that (4.3) holds for all $p \in (p_0 - \delta_2, p_0 + \delta_2)$, and take $\delta = \min(\delta_1, \delta_2)$.

Fix $r = r_c(p_0) + \varepsilon$, and define the event $V_n = \{(\xi, \eta) \in \Omega_{\mathbb{Z}^2} \times \Omega_{\mathcal{E}_2}$: there exists a vertical crossing of $[0, n] \times [0, 3n]$ that is black in $\xi\}$. By "vertical crossing," we mean a self-avoiding path of vertices in $[0, n] \times [0, 3n]$ with one endpoint in $[0, n] \times \{0\}$, and one in $[0, n] \times \{3n\}$. Recall also the definition of M_n in Theorem 2.1. By Lemma 2.10 in Bálint et al. (2009), there exists a constant $\gamma > 0$ such that the following implication holds for any $p, a \in [0, 1]$ and $L \in \mathbb{N}$:

$$\left.\begin{array}{rcl} (3L+1)(L+1)\nu_a(M_{\lfloor L/3 \rfloor}) & \leq & \gamma, \\ \text{and } \mathbb{P}_{p,a}(V_L) & \geq & 1-\gamma \end{array}\right\} \Rightarrow a \geq r_c(p).$$

As usual, $\lfloor x \rfloor$ for $x > 0$ denotes the largest integer m such that $m \leq x$. Fix such a γ.

By Theorem 2.1, there exists $N \in \mathbb{N}$ such that

$$(3n+1)(n+1)\nu_{p_0}(M_{\lfloor n/3 \rfloor}) < \gamma$$

for all $n \geq N$. On the other hand, since $r > r_c(p_0)$, it follows from Lemma 2.11 in Bálint et al. (2009) that there exists $L \geq N$ such that

$$\mathbb{P}_{p_0,r}(V_L) > 1 - \gamma.$$

Note that both $(3L + 1)(L + 1)\nu_p(M_{\lfloor L/3 \rfloor})$ and $\mathbb{P}_{p,r}(V_L)$ are continuous in p at p_0. Indeed, the former is simply a polynomial in p, while the continuity of the latter follows from Proposition 4.1. Therefore, there exists $\delta_1 > 0$ such that for all $p \in (p_0 - \delta_1, p_0 + \delta_1)$,

$$\begin{array}{rcl} (3L+1)(L+1)\nu_p(M_{\lfloor L/3 \rfloor}) & \leq & \gamma, \\ \text{and } \mathbb{P}_{p,r}(V_L) & \geq & 1-\gamma. \end{array}$$

By the choice of γ, this implies that $r \geq r_c(p)$ for all such p, which is precisely what we wanted to prove.

Finding $\delta_2 > 0$ such that (4.3) holds for all $p \in (p_0 - \delta_2, p_0 + \delta_2)$ is analogous: one only needs to substitute $r_c(p_0)$ by $r_c^*(p_0)$ and "crossing" by "$*$-crossing," and the exact same argument as above works. It follows that $\delta = \min(\delta_1, \delta_2) > 0$ is a constant such that both (4.2) and (4.3) hold for all $p \in (p_0 - \delta, p_0 + \delta)$, completing the proof of continuity on $(0, 1/2)$. Right-continuity at 0 may be proved analogously; alternatively, it follows from Proposition 1.2. □

Remark 4.4. It follows from Theorem 1.6 and equation (2.1) that $r_c^*(p)$ is also continuous in p on $[0, 1/2)$.

5. The critical value functions of tree-like graphs

In this section, we will study the critical value functions of graphs that are constructed by replacing edges of an infinite tree by a sequence of finite graphs. We will then use several such constructions in the proofs of our main results in Section 6.

Let us fix an arbitrary sequence $D_n = (\mathcal{V}_n, \mathcal{E}_n)$ of finite connected graphs and, for every $n \in \mathbb{N}$, two distinct vertices $a_n, b_n \in \mathcal{V}_n$. Let $\mathbb{T}_3 = (V_3, E_3)$ denote the (infinite) regular tree of degree 3, and fix an arbitrary vertex $\rho \in V_3$. Then, for each edge $e \in E_3$, we denote the end-vertex of e which is closer to ρ by $f(e)$, and the other end-vertex by $s(e)$. Let $\Gamma_D = (\tilde{V}, \tilde{E})$ be the graph obtained by replacing every edge e of Γ_3 between levels $n-1$ and n (*i.e.*, such that $dist(s(e), \rho) = n$) by a copy D_e of D_n, with a_n and b_n replacing respectively $f(e)$ and $s(e)$. Each vertex $v \in V_3$ is replaced by a new vertex in $\tilde{V}$, which we denote by $\tilde{v}$. It is well known that $p_c^{\Gamma_3} = r_c^{\Gamma_3}(0) = 1/2$. Using this fact and the tree-like structure of Γ_D, we will be able to determine bounds for $p_c^{\Gamma_D}$ and $r_c^{\Gamma_D}(p)$.

First, we define $h^{D_n}(p) = \nu_p^{\mathcal{E}_n}(a_n \text{ and } b_n \text{ are in the same bond cluster})$, and prove the following, intuitively clear, lemma.

Lemma 5.1. *For any $p \in [0, 1]$, the following implications hold:*

a) *if* $\limsup_{n\to\infty} h^{D_n}(p) < 1/2$, *then* $p \le p_c^{\Gamma_D}$;
b) *if* $\liminf_{n\to\infty} h^{D_n}(p) > 1/2$, *then* $p \ge p_c^{\Gamma_D}$.

Proof. We couple Bernoulli bond percolation with parameter p on Γ_D with inhomogeneous Bernoulli bond percolation with parameters $h^{D_n}(p)$ on $\mathbb{T}_3$, as follows. Let η be a random variable with law $\nu_p^{\tilde{E}}$, and define, for each edge $e \in E_3$, $W(e) = 1$ if $\widetilde{f(e)}$ and $\widetilde{s(e)}$ are connected by a path consisting of edges that are open in η, and $W(e) = 0$ otherwise. The tree-like structure of Γ_D implies that $W(e)$ depends only on the state of the edges in D_e, and it is clear that if $dist(s(e), \rho) = n$, then $W(e) = 1$ with probability $h^{D_n}(p)$.

It is easy to verify that there exists an infinite open self-avoiding path on Γ_D from $\tilde{\rho}$ in the configuration η if and only if there exists an infinite open self-avoiding path on $\mathbb{T}_3$ from ρ in the configuration W. Now, if we assume $\limsup_{n\to\infty} h^{D_n}(p) < 1/2$, then there exists $t < 1/2$ and $N \in \mathbb{N}$ such that for all $n \ge N$, $h^{D_n}(p) \le t$. Therefore, the distribution of the restriction of W on $L = \{e \in E_3 : dist(s(e), \rho) \ge N\}$ is stochastically dominated by the projection of $\nu_t^{E_3}$ on L. This implies that, a.s., there exists no infinite self-avoiding path in W, whence $p \le p_c^{\Gamma_D}$ by the observation at the beginning of this paragraph. The proof of b) is analogous. □

We now turn to the DaC model on Γ_D. Recall that for a vertex v, C_v denotes the vertex set of the bond cluster of v. Let $E_{a_n,b_n} \subset \Omega_{\mathcal{E}_n} \times \Omega_{\mathcal{V}_n}$ denote the event that a_n and b_n are in the same bond cluster, or a_n and b_n lie in two different bond clusters, but there exists a vertex v at distance 1 from C_{a_n} which is connected to b_n by a black path (which also includes that $\xi(v) = \xi(b_n) = 1$). This is the same as saying that C_{a_n} is *pivotal* for the event that there is a black path between a_n and b_n, *i.e.*, that such a path exists if and only if C_{a_n} is black. It is important to note that E_{a_n,b_n} is independent of the color of a_n. Define $f^{D_n}(p, r) = \mathbb{P}_{p,r}^{D_n}(E_{a_n,b_n})$, and note also that, for $r > 0$, $f^{D_n}(p, r) = \mathbb{P}_{p,r}^{D_n}(\text{there is a black path from } a_n \text{ to } b_n \mid \xi(a_n) = 1)$.

Lemma 5.2. *For any $p, r \in [0, 1]$, we have the following:*

a) *if* $\limsup_{n\to\infty} f^{D_n}(p, r) < 1/2$, *then* $r \le r_c^{\Gamma_D}(p)$;
b) *if* $\liminf_{n\to\infty} f^{D_n}(p, r) > 1/2$, *then* $r \ge r_c^{\Gamma_D}(p)$.

Proof. We couple here the DaC model on Γ_D with inhomogeneous Bernoulli *site* percolation on $\mathbb{T}_3$. For each $v \in V_3 \setminus \{\rho\}$, there is a unique edge $e \in E_3$ such that $v = s(e)$. Here we denote D_e (*i.e.*, the subgraph of Γ_D replacing the edge e) by $D_{\tilde{v}}$, and the analogous event of E_{a_n,b_n} for the graph $D_{\tilde{v}}$ by $E_{\tilde{v}}$. Let (η, ξ) with values in $\Omega_{\tilde{E}} \times \Omega_{\tilde{V}}$ be a random variable with law $\mathbb{P}_{p,r}^{\Gamma_D}$. We define a random variable X

with values in Ω_{V_3}, as follows:

$$X(v) = \begin{cases} \xi(\tilde{\rho}) & \text{if } v = \rho, \\ 1 & \text{if the event } E_{\tilde{v}} \text{ is realized by the restriction of } (\eta, \xi) \text{ to } D_{\tilde{v}}, \\ 0 & \text{otherwise.} \end{cases}$$

As noted after the proof of Lemma 5.1, if $u = f(\langle u, v\rangle)$, the event $E_{\tilde{v}}$ is independent of the color of $\tilde{u}$, whence $(E_{\tilde{v}})_{v\in V_3\setminus\{\rho\}}$ are independent. Therefore, as $X(\rho) = 1$ with probability r, and $X(v) = 1$ is realized with probability $f^{D_n}(p, r)$ for $v \in V_3$ with $dist(v, \rho) = n$ for some $n \in \mathbb{N}$, X is inhomogeneous Bernoulli site percolation on $\mathbb{T}_3$.

Our reason for defining X is the following property: it holds for all $v \in V_3 \setminus \{\rho\}$ that

$$\tilde{\rho} \overset{\xi}{\leftrightarrow} \tilde{v} \quad \text{if and only if} \quad \rho \overset{X}{\leftrightarrow} v, \tag{5.1}$$

where $x \overset{Z}{\leftrightarrow} y$ denotes that x and y are in the same *black* cluster in the configuration Z. Indeed, assuming $\tilde{\rho} \overset{\xi}{\leftrightarrow} \tilde{v}$, there exists a path $\rho = x_0, x_1, \cdots, x_k = v$ in Γ_3 such that, for all $0 \le i < k$, $\tilde{x_i} \overset{\xi}{\leftrightarrow} \tilde{x_{i+1}}$ holds. This implies that $\xi(\tilde{\rho}) = 1$ and that all the events $(E_{\tilde{x_i}})_{0<i\le k}$ occur, whence $X(x_i) = 1$ for $i = 0, \ldots, k$, so $\rho \overset{X}{\leftrightarrow} v$ is realized. The proof of the other implication is similar. It follows in particular from (5.1) that $\tilde{\rho}$ lies in an infinite black cluster in the configuration ξ if and only if ρ lies in an infinite black cluster in the configuration X.

Lemma 5.2 presents two scenarios when it is easy to determine (via a stochastic comparison) whether the latter event has positive probability. For example, if we assume that $\liminf_{n\to\infty} f^{D_n}(p, r) > 1/2$, then there exists $t > 1/2$ and $N \in \mathbb{N}$ such that for all $n \ge N$, $f^{D_n}(p, r) \ge t$. In this case, the distribution of the restriction of X on $K = \{v \in V_3 : dist(v, \rho) \ge N\}$ is stochastically larger than the projection of $\nu_t^{E_3}$ on K. Let us further assume that $r > 0$. In that case, $X(\rho) = 1$ with positive probability, and $f^{D_n}(p, r) > 0$ for every $n \in \mathbb{N}$. Therefore, under the assumptions $\liminf_{n\to\infty} f^{D_n}(p, r) > 1/2$ and $r > 0$, ρ is in an infinite black cluster in X (and, hence, $\tilde{\rho}$ is in an infinite black cluster in ξ) with positive probability, which can only happen if $r \ge r_c^{\Gamma_D}(p)$. On the other hand, if $\liminf_{n\to\infty} f^{D_n}(p, 0) > 1/2$, then it is clear that $\liminf_{n\to\infty} f^{D_n}(p, r) > 1/2$ (whence $r \ge r_c^{\Gamma_D}(p)$) for all $r > 0$, which implies that $r_c^{\Gamma_D}(p) = 0$. The proof of part a) is similar. □

6. Counterexamples

In this section, we study two particular graph families and obtain examples of non-monotonicity and non-continuity of the critical value function.

6.1. *Non-monotonicity.* The results in Section 5 enable us to prove that (a small modification of) the construction considered by Häggström in the proof of Theorem 2.9 in Häggström (2001) is a graph whose critical coloring value is non-monotone in the subcritical phase.

Proof of Proposition 1.7. Define for $k \in \mathbb{N}$, D^k to be the complete bipartite graph with the vertex set partitioned into $\{z_1, z_2\}$ and $\{a, b, v_1, v_2, \ldots, v_k\}$ (see Figure 6.1). We call e_1, e_1' and e_2, e_2' the edges incident to a and b respectively, and for $i = 1, \ldots, k$, f_i, f_i' the edges incident to v_i. Consider Γ_k the quasi-transitive graph obtained by replacing each edge of the tree $\mathbb{T}_3$ by a copy of D_k. Γ_k can be seen as the tree-like graph resulting from the construction described at beginning of the section, when we start with the constant sequence $(D_n, a_n, b_n) = (D^k, a, b)$.

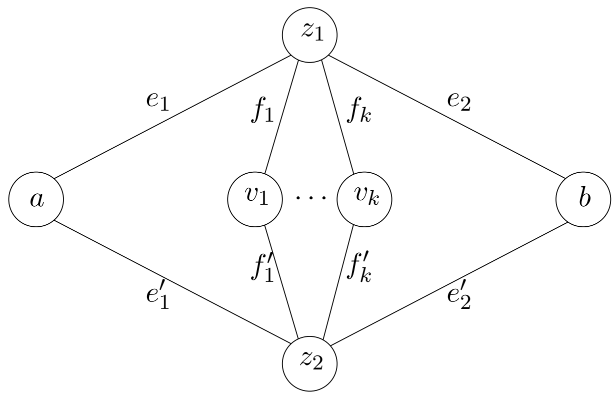

FIGURE 6.1. The graph D^k.

We will show below that it holds for all $k \in \mathbb{N}$ that

$$p_c^{\Gamma_k} > 1/3, \tag{6.1}$$

$$r_c^{\Gamma_k}(0) < 2/3, \quad \text{and} \tag{6.2}$$

$$r_c^{\Gamma_k}(1/3) < 2/3. \tag{6.3}$$

Furthermore, there exists $k \in \mathbb{N}$ and $p_0 \in (0, 1/3)$ such that

$$r_c^{\Gamma_k}(p_0) > 2/3. \tag{6.4}$$

Proving (6.1)–(6.4) will finish the proof of Proposition 1.7 since these inequalities imply that the quasi-transitive graph Γ_k has a non-monotone critical value function in the subcritical regime.

Throughout this proof, we will omit superscripts in the notation when no confusion is possible. For the proof of (6.1), recall that h^{D^k} is strictly increasing in p, and $h^{D^k}(p_{D^k}) = 1/2$. Since $1 - h^{D^k}(p)$ is the ν_p-probability of a and b being in two different bond clusters, we have that

$$1 - h^{D^k}(1/3) \geq \nu_{1/3}(\{e_1 \text{ and } e_1' \text{ are closed}\} \cup \{e_2 \text{ and } e_2' \text{ are closed}\}).$$

From this, we get that $h^{D^k}(1/3) \leq 25/81$, which proves (6.1).

To get (6.2), we need to remember that for fixed $p < p_{D^k}$, $f^{D^k}(p, r)$ is strictly increasing in r, and $f^{D^k}(p, r_{D^k}(p)) = 1/2$. One then easily computes that $f(0, 2/3) = 16/27 > 1/2$, whence (6.2) follows from Lemma 5.2.

Now, define A to be the event that at least one edge out of e_1, e_1', e_2 and e_2' is open. Then

$$\begin{aligned} f^{D^k}(1/3, 2/3) &\geq \mathbb{P}_{1/3,2/3}(E_{a,b} \mid A)\mathbb{P}_{1/3,2/3}(A) \\ &\geq \mathbb{P}_{1/3,2/3}(C_b \text{ black} \mid A) \cdot 65/81, \end{aligned}$$

which gives that $f^{D^k}(1/3, 2/3) \geq 130/243 > 1/2$, and implies (6.3) by 5.2.

To prove (6.4), we consider B_k to be the event that e_1, e_1', e_2 and e_2' are all closed and that there exists i such that f_i and f_i' are both open. One can easily compute that

$$\mathbb{P}_{p,r}(B_k) = (1-p)^4 \left(1 - (1-p^2)^k\right),$$

which implies that we can choose $p_0 \in (0, 1/3)$ (small) and $k \in \mathbb{N}$ (large) such that $\mathbb{P}_{p_0,r}(B_k) > 17/18$. Then,

$$\begin{aligned} f^{D^k}(p_0, 2/3) &= \mathbb{P}_{p_0,r}(E_{a,b} \mid B_k)\mathbb{P}_{p_0,r}(B_k) + \mathbb{P}_{p_0,r}(E_{a,b} \mid B_k^c)(1-\mathbb{P}_{p_0,r}(B_k)) \\ &< (2/3)^2 \cdot 1 + 1 \cdot 1/18 (= 1/2), \end{aligned}$$

whence inequality (6.4) follows with these choices from Lemma 5.2, completing the proof. □

6.2. *Graphs with discontinuous critical value functions.*
Proof of Proposition 1.3. For $n \in \mathbb{N}$, let D_n be the graph depicted in Figure 6.2, and let G be Γ_D constructed with this sequence of graphs as described at the beginning of Section 5.

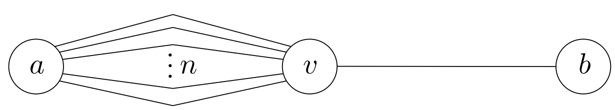

FIGURE 6.2. The graph D_n.

It is elementary that $\lim_{n\to\infty} h^{D_n}(p) = p$, whence $p_c^G = 1/2$ follows from Lemma 5.1, thus $p = 0$ is subcritical. Since $\lim_{n\to\infty} f^{D_n}(0, r) = r^2$, Lemma 5.2 gives that $r_c^G(0) = 1/\sqrt{2}$. On the other hand, $\lim_{n\to\infty} f^{D_n}(p, r) = p + (1-p)r$ for all $p > 0$, which implies by Lemma 5.2 that for $p \leq 1/2$,

$$r_c^G(p) = \frac{1/2 - p}{1-p} \to 1/2$$

as $p \to 0$, so r_c^G is indeed discontinuous at $0 < p_c^G$. □

In the rest of this section, for vertices v and w, we will write $v \leftrightarrow w$ to denote that there exists a path of open edges between v and w. Our proof of Theorem 1.4 will be based on the Lemma 2.1 in Peres et al. (2009), that we rewrite here:

Lemma 6.1. *There exists a sequence $G_n = (V^n, E^n)$ of graphs and $x_n, y_n \in V^n$ of vertices ($n \in \mathbb{N}$) such that*

1. $\nu_{1/2}^{E^n}(x_n \leftrightarrow y_n) > \frac{2}{3}$ *for all n;*
2. $\lim_{n\to\infty} \nu_p^{E^n}(x_n \leftrightarrow y_n) = 0$ *for all $p < 1/2$, and*
3. *there exists $\Delta < \infty$ such that, for all n, G_n has degree at most Δ.*

Lemma 6.1 provides a sequence of bounded degree graphs that exhibit sharp threshold-type behavior at 1/2. We will use such a sequence as a building block to obtain discontinuity at 1/2 in the critical value function in the DaC model.

Proof of Theorem 1.4. We first prove the theorem in the case $p_0 = 1/2$. Consider the graph $G_n = (V^n, E^n), x_n, y_n$ ($n \in \mathbb{N}$) as in Lemma 6.1. We construct D_n from G_n by adding to it one extra vertex a_n and one edge $\{a_n, x_n\}$. More precisely D_n has vertex set $V^n \cup \{a_n\}$ and edge set $E^n \cup \{a_n, x_n\}$. Set $b_n = y_n$ and let G be the graph Γ_D defined with the sequence (D_n, a_n, b_n) as in Section 5.

We will show below that there exists $r_0 > r_1$ such that the graph G verify the following three properties:

(i) $1/2 < p_c^G$
(ii) $r_c^G(p) \geq r_0$ for all $p < 1/2$.
(iii) $r_c^G(1/2) \leq r_1$.

It implies a discontinuity of r_c^G at $1/2 < p_c^G$, finishing the proof.

One can easily compute $h^{D_n}(p) = p\nu_p^{E^n}(x_n \leftrightarrow y_n)$. Since the graph G_n has degree at most Δ and the two vertices x_n, y_n are disjoint, the probability $\nu_p^{E^n}(x_n \leftrightarrow y_n)$ cannot exceed $1 - (1-p)^\Delta$. This bound guarantees the existence of $p_0 > 1/2$ independent of n such that $h^{D_n}(p_0) < 1/2$ for all n, whence Lemma 5.1 implies that $1/2 < p_0 \leq p_c^G$.

For all $p \in [0, 1]$, we have

$$f^{D_n}(p, r) \leq (p + r(1-p)) \left(\nu_p^{E^n}(x_n \leftrightarrow y_n) + r(1 - \nu_p^{E^n}(x_n \leftrightarrow y_n)) \right)$$

which gives that $\lim_{n\to\infty} f^{D_n}(p, r) < \left(\frac{r+1}{2}\right) r$. Denoting by r_0 the positive solution of $r(1+r) = 1$, we get that $\lim_{n\to\infty} f^{D_n}(p, r_0) < 1/2$ for all $p < 1/2$, which implies by Lemma 5.2 that $r_c^G(p) \geq r_0$.

On the other hand, $f^{D_n}(1/2, r) \geq \nu_p^{E^n}(x_n \leftrightarrow y_n)\left(\frac{1+r}{2}\right)$, which gives by Lemma 6.1 that $\lim_{n\to\infty} f^{D_n}(1/2, r) > \frac{2}{3} \cdot \frac{1+r}{2}$. Writing r_1 such that $\frac{2}{3}(1 + r_1) = 1$, it is elementary to check that $r_1 < r_0$ and that $\lim_{n\to\infty} f^{D_n}(1/2, r_1) > 1/2$. Then, using Lemma 5.2, we conclude that $r_c(1/2) \leq r_1$. □

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