TREES OF SELF-AVOIDING WALKS

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Abstract. We consider the biased random walk on a tree constructed from the set of finite self-avoiding walks on a lattice, and use it to construct probability measures on infinite self-avoiding walks. The limit measure (if it exists) obtained when the bias converges to its critical value is conjectured to coincide with the weak limit of the uniform SAW. Along the way, we obtain a criterion for the continuity of the escape probability of a biased random walk on tree as a function of the bias, and show that the collection of escape probability functions for spherically symmetric trees of bounded degree is stable under uniform convergence.

Keywords: Self-avoiding walk, equivalent conductance, random walk on tree

1. Introduction

An n-step self-avoiding walk (SAW) (or a self-avoiding walk of length n) in a regular lattice $\mathbb{L}$ (such as integer lattice $\mathbb{Z}^2$, triangular lattice $\mathbb{T}$, hexagonal lattice, etc) starting at the origin is a nearest neighbor path $\gamma = [\gamma_0 = 0, \gamma_1, \ldots, \gamma_n]$ that visits no vertex more than once. An infinite self-avoiding walk is a self-avoiding walk of infinite length. Self-avoiding walks were first introduced as a lattice model for polymer chains; while they are very easy to define, they are extremely difficult to analyze rigorously and there are still many basic open questions about them.

Let c_n be the number of SAWs of length n starting at the origin. The *connective constant* of $\mathbb{L}$, which we will denote by μ, is defined by

$$c_n \approx \mu^n \text{ when } n \to \infty.$$

The existence of the connective constant is easy to establish from the sub-additivity relation $c_{n+m} \leqslant c_n c_m$, from which one can also deduce that $c_n \geqslant \mu^n$ for all n. Nienhuis [16] gave a prediction that for all regular plan lattices, $c_n = \mu^n n^{\alpha + o(1)}$ where $\alpha = \frac{11}{32}$, and this prediction is known to hold under the assumption of the existence of a conformally invariant scaling limit, see *e.g.* [11].

We are interested in defining a natural probability measure on the set of *infinite* self-avoiding walks (SAW_∞) (see the sections 5.2 and 6). Such a measure on the set of the infinite self-avoiding half-plane walks has been constructed already as the weak limit of the uniform measures on the finite self-avoiding walks (see [14]) by using the Kesten's equality [9].

In this paper, we consider a one-parameter family of probability measures on SAW_∞, denoted by $(\mathbb{P}_\lambda)_{\lambda > \lambda_c}$, defined informally as follows. Denote by $\mathbb{H}$ the upper-half plane in $\mathbb{Z}^2$ and by $\mathbb{Q}$ the first quadrant; let $T_{\mathbb{Z}^2}$ (resp. $T_\mathbb{H}$, $T_\mathbb{Q}$, with the appropriate modifications in the definition which we will not specify in what follows) be the tree whose vertices are the finite self-avoiding walks in the plane (respectively half-plane, quadrant), where two such vertices are adjacent when one walk is a one-step extension of the other. We will call this tree the *self-avoiding tree* on $\mathbb{Z}^2$.

Then, consider the continuous-time biased random walk of parameter $\lambda > 0$ on $T_{\mathbb{Z}^2}$, which from a given location jumps towards the root with rate 1 and towards each of its children vertices with rate λ. If λ is such that the walk is transient, its path determines an infinite branch in $T_{\mathbb{Z}^2}$ which can be seen as a random infinite self-avoiding walk ω_λ^∞; we will denote by $\mathbb{P}_\lambda$ the law of ω_λ^∞, omitting the mention of $\mathbb{Z}^2$ in the notation, and call it the **limit walk** with parameter λ.

It is well known that there exists a critical value λ_c such that if $\lambda > \lambda_c$ the biased random walk is transient and if $\lambda < \lambda_c$ it is recurrent. In the general case of biased random walk on a tree, the recurrence or transience of the random walk at the critical point depends in subtle ways on the structure of the tree. The value of λ_c on the other hand is easier to determine:

indeed, Lyons [12] proved that it coincides with the reciprocal of the branching rate of the tree. The following proposition give the critical value for self-avoiding trees.

Proposition 1.1. *Let $T_{\mathbb{Z}^2}, T_{\mathbb{H}}, T_{\mathbb{Q}}$ be defined as above. Then,*

$$\lambda_c(T_{\mathbb{Z}^2}) = \lambda_c(T_{\mathbb{H}}) = \lambda_c(T_{\mathbb{Q}}) = \frac{1}{\mu},$$

where μ is the connective constant of lattice $\mathbb{Z}^2$.

This is a direct consequence of Proposition 5.10 below. Notice that it is clear from the definition that μ is the growth rate of $T_{\mathbb{Z}^2}$; there are rather large classes of trees, including $T_{\mathbb{Z}^2}$, for which the branching and growth coincide (for instance, this holds for sub- or super-periodic trees, cf. below, or for typical supercritical Galton-Watson trees), but none of the classical results seem to apply to $T_{\mathbb{H}}$ or $T_{\mathbb{Q}}$.

We now state some properties concerned with the geometry of the limit walk for this family of probability measures.

Theorem 1.2. *For all $\lambda > \lambda_c$, under the $\mathbb{P}_\lambda$ measure, the infinite self-avoiding walk (in the plane or half-plane) reaches the line $\mathbb{Z} \times \{0\}$ infinitely many times almost surely.*

Theorem 1.3. *For all $\lambda > \lambda_c$, then*

$$\mathbb{P}_\lambda(\limsup_n \Re\omega_\lambda^\infty(n) = +\infty) = 1; \quad \mathbb{P}_\lambda(\liminf_n \Re\omega_\lambda^\infty(n) = -\infty) = 1.$$

These theorems are proved in Section 6.4. We are mostly interested in the behavior of the limit walk as $\lambda \to \lambda_c$, since this is a natural candidate to be in relation with uniformly sampled long SAWs. We did not quite manage to prove the existence of the limit, but were able to obtain a partial result in this direction by restricting the process to paths formed of bridges of bounded height m, and letting m increase; see Theorem 7.3 for more details.

A useful tool in our proofs is the **effective conductance** of the biased random walk on a tree T, defined as the probability of never returning to the root o of T and denoted by $C(\lambda, T)$. Along the way, we will be interested in several properties of it as a function of λ. Most important for us will be the question of continuity: in a general tree, the effective conductance is not necessarily a continuous function of λ. We will derive criteria for continuity, which are forms of **uniform transience** of the random walk, and apply them to prove that the effective conductance of self-avoiding trees is a continuous function (see Section 5.5):

Theorem 1.4. *The functions $C(\lambda, T_{\mathbb{H}})$ and $C(\lambda, T_{\mathbb{Z}^2})$ are continuous on $(\lambda_c, +\infty)$.*

A related question is that of the convergence of effective conductance along a sequence of trees. More precisely, let $(f_n)_n$ denote the effective conductances for a sequence (T_n) of infinite trees, and we assume that $(f_n)_n$ converges uniformly towards $f \neq 0$. The question is: is f the effective conductance of a certain tree? We study this question on a class of particular trees, spherically symmetric trees (recall that T is spherically symmetric if $\deg x$ depends only on $|x|$, where $|x|$ denote its distance from the root o and $\deg x$ is the number of its neighbors). If $\mathbb{S}$ denotes the set of spherically symmetric trees and $m \in \mathbb{N}^*$ is fixed, define

$$A_m := \{T \in \mathbb{S}; \forall x \in T, \deg x \leqslant m\} \text{ and } \mathbb{F}_m := \left\{f \in C^0([0,1]) : \exists T \in A_m, C(\lambda, T) = f(\lambda)\right\}.$$

Then (see Section 4.2):

Theorem 1.5. *Let $(f_n)_n$ be a sequence of functions in $\mathbb{F}_m$. Assume that f_n converges uniformly towards $f \neq 0$. Then $f \in \mathbb{F}_m$.*

The paper is structured as follows. In Section 2, we review some basic definitions on graphs, tree, branching number and growth rate of a tree, as well as a few classical results about random walks on trees. Section 3 gathers some relevant examples and counter-examples exhibiting some similarities to the self-avoiding trees while being treatable explicitly. The criterion for the continuity of the effective conductance is given in Section 4. Then Section 5 provides some background on self-avoiding walks and the self-avoiding trees, and some properties of the limit walks are obtained in Section 6. Finally, we state a few conjectures and conditional results in Section 7. Appendix A isolates the details of an algorithm to improve the readability of the main text.

2. Notation and basic definitions

2.1. **Graphs and trees.** In this section, we review some basic definitions; we refer the reader to the book [13] for a more developed treatment. A *graph* is a pair $G = (V, E)$ where V is a set of *vertices* and E is a symmetric subset of $V \times V$, called the *edge set*, containing no element of the form (x, x). Two vertices are *adjacent* if their pair that they form is an edge. A *path* in a graph is a sequence of vertices, any two consecutive of which are adjacent. A *simple path* is a path which does not pass through any vertex more than once. A graph is *connected* if, for each pair $(x, y) \in V \times V$, there exist a simple path starting at x and ending at y. A connected graph with no cycles is called a *tree*. We will always consider trees to be *rooted* by the choice of a vertex o, called the *root*.

Let T be a rooted tree and $x \in V(T)$, the symbol $|x|$ will denote the *height* of x, that is the distance from x to o in the graph distance, *i.e.* the length of the simple path joining o to this vertex; $\deg x$ will denote the number of neighbors of x. Let T_n be the set of vertices of T with height n. The *parent* of a vertex is the vertex connected to it on the simple path to the root; every vertex except the root has a unique parent. A *child* of a vertex v is a vertex of which v is the parent. A vertex is called a *leaf* if it have no child. We define an order on $V(T)$ as follows: if $x, y \in V(T)$, we say that $x \leqslant y$ if the simple path joining o to y passes through x. For each $x \in T$, we define the *sub-tree* T^x where $V(T^x) := \{y \in T : x \leqslant y\}$ and $E(T^x) = E(T)|_{V(T^x)\times V(T^x)}$.

An infinite simple path starting at o is called a *ray*. The set of all rays, denoted by ∂T, is called the *boundary* of T. The set $T \cup \partial T$ can be equipped with a metric that makes it a compact space, see [13].

2.2. **Branching and growth.**

Definition 2.1. Given a graph $G = (V, E)$ and A, Z two subsets of V, a set $\Pi \subset V$ is said to *separate A and Z* (or to be a *cut-set between A and B*) if every path starting at a point in A and finishing at a point in Z must pass through a vertex of Π. Similarly, if G is infinite and equipped with a marked root o, Π is said to *separate o and ∞* if every infinite simple path started from o must pass through a vertex of Π; we also call Π a cut-set. For example, let T be a tree, then for all n, T_n is a cut-set of T.

Definition 2.2. Let T be a tree.

- The *branching number* of T is defined by:
$$br(T) = \sup\left\{\lambda \geqslant 1 : \inf_{\Pi} \sum_{e\in\Pi} \lambda^{-|e|} > 0\right\},$$
where the inf is taken over cut-sets of T.
- We define also
$$\overline{gr}(T) = \limsup |T_n|^{1/n} \quad \text{and} \quad \underline{gr}(T) = \liminf |T_n|^{1/n}.$$
In the case $\overline{gr}(T) = \underline{gr}(T)$, the *growth rate* of T is defined by their common value and denoted by $gr(T)$.

Proposition 2.3 ([13])**.** *Let T be a tree, then $br(T) \leqslant \underline{gr}(T)$.*

In general, the inequality in Proposition 2.3 may be strict: The *1–3 tree* (see [13], page 4) is an example for which the branching number is 1 and the growth rate is 2. There are classes of trees however where branching and growth match.

Definition 2.4. The tree T is said to be *spherically symmetric* if $\deg x$ depends only on $|x|$.

Theorem 2.5 ([13] page 83)**.** *For every spherically symmetric tree T, $br(T) = \underline{gr}(T)$.*

Definition 2.6. Let $N \geqslant 0$: an infinite tree T is said to be

- *N-sub-periodic* if for every $x \in T$, there exists an adjacency-preserving injection $f : T^x \to T^{f(x)}$ with $|f(x)| \leqslant N$.
- *N-super-periodic* if for every $x \in T$, there exists an adjacency-preserving injection $f : T \to T^{f(o)}$ with $d(x, f(o)) \leqslant N$.

Theorem 2.7 (see [5, 13])**.** *Let T be a tree that is either N-sub-periodic, or N-super-periodic with $\overline{gr}(T) < \infty$. Then the growth rate of T exists and $gr(T) = br(T)$.*

2.3. **Random walks on trees.** Let T be a tree, we now define the discrete-time biased random walk on T. Working in discrete time will make some of the arguments below a little simpler, at the cost of a slightly heavier definition here — notice though that the definition of the measure $\mathbb{P}_\lambda$ and the main results of the paper are not at all affected by this choice.

Let $\lambda > 0$: the biased walk RW_λ with bias λ on T is the discrete-time Markov chain on the vertex set ot T with transition probabilities given, at a vertex $x \neq o$ with k children, by

$$p_\lambda(x,y) := \begin{cases} \frac{1}{1+k\lambda} & \text{if } y \text{ is the father of } x, \\ \frac{\lambda}{1+k\lambda} & \text{if } y \text{ is a child of } x, \\ 0 & \text{otherwise.} \end{cases}$$

If the root has $k > 0$ children, then $p_\lambda(o,x)$ is $1/k$ if x is a child of o and 0 otherwise. The degenerate case $T = \{o\}$ where the root has no child will not occur in our context, so we will silently ignore it. We also allow ourselves to consider the cases $\lambda \in \{0, \infty\}$, with the natural convention that RW_0 remains stuck at the root.

Definition 2.8. Let $G = (V,E)$ be a graph, and $c : E \to \mathbb{R}_+^*$ be labels on the edges, referred to as *conductances*. Equivalently, one can fix *resistances* by letting $r(e) := 1/c(e)$. The pair (G,c) is called a *network*. Given a subset K of V, the restriction of c to the edges joining vertices in K defines the *induced sub-network* $G|_K$. The *random walk* on the network (G,c) is the discrete-time Markov chain on V with transition probabilities proportional to the conductances.

Given a network (T,e) on a tree, let $\pi(o)$ be the sum of the conductances of the edges incident to the root, and denote by $o \to \infty$ the event that the random walk on (T,e), started at the root, never returns to it. We will write $\widetilde{C}(o \leftrightarrow \infty) := \mathbb{P}[o \to \infty]$ and $C(o \leftrightarrow \infty) := \pi(o)\widetilde{C}(o \leftrightarrow \infty)$. The latter is the *equivalent conductance* of the network, and its reciprocal $R(o \leftrightarrow \infty)$ is the *equivalent resistance*.

The particular case where, on a tree T, for an edge $e = (x,y)$ between a vertex x and any of its children y, $c(e)$ is chosen to be $\lambda^{|x|}$ will play a special role, because in that case the random walk on the network is exactly the same process as the random walk RW_λ defined earlier. Is this setup, we will denote the equivalent conductance by $C(\lambda, T)$ to emphasize its dependency on the parameter λ.

Theorem 2.9 (Rayleigh's monotonicity principle [13])**.** *Let T be an infinite tree with two assignments, c and c', of conductances on T with $c \leqslant c'$ (everywhere). Then the equivalent conductances are ordered in the same way: $C_c(o \leftrightarrow \infty) \leqslant C_{\tilde{c}}(o \leftrightarrow \infty)$.*

Corollary 2.10. *Let T, T' be two infinite trees; we say that $T \subset T'$ if there exists an adjacency-preserving injection $f : T \to T'$. If this holds, then for every $\lambda > 0$, $C(\lambda, T') \leqslant C(\lambda, T)$.*

In the case of spherically symmetric trees, the equivalent resistance is explicit:

Proposition 2.11. *Let T be spherically symmetric and $(c(e))$ be conductances that are themselves constant on the levels of T. Then $R(o \leftrightarrow \infty) = \sum_{n \geqslant 1} \frac{1}{c_n |T_n|}$, where c_n is the conductance of the edges going from level $n-1$ to level n.*

Corollary 2.12. *Let T be a spherically symmetric tree. Then RW_λ is transient if and only if $\sum_n \frac{1}{\lambda^n |T_n|} < \infty$.*

Theorem 2.13 (Nash-Williams inequality, see [15]). *If a and z are distinct vertices in a finite network that are separate by pairwise disjoint cut-sets $\Pi_1, \Pi_2, \ldots, \Pi_n$, then*

$$R(a \leftrightarrow z) \geqslant \sum_{k=1}^{n} (\sum_{e^- \in \Pi_k} c(e))^{-1}.$$

This theorem implies the following theorem

Theorem 2.14 (Nash-Williams criterion, see [15]). *If Π_n is a sequence of pairwise disjoint finite cut-sets in a locally finite network G, then*

$$R(o \leftrightarrow \infty) \geqslant \sum_n \left(\sum_{e^- \in \Pi_n} c(e) \right)^{-1}.$$

In particular, if $\sum_n \left(\sum_{e \in \Pi_n} c(e)\right)^{-1} = +\infty$, then the random walk associated to this family conductances $(c(e))_e$ is recurrent.

We end this subsection by stating a classical theorem relating the recurrence or transience of RW_λ to the branching of the underlying tree:

Theorem 2.15 (see [12]). *If $\lambda < \frac{1}{br(T)}$ then RW_λ is recurrent, whereas if $\lambda > \frac{1}{br(T)}$, then RW_λ is transient. The critical value of biased random walk on T is therefore $\lambda_c(T) := \frac{1}{br(T)}$.*

2.4. **The law of first k-steps of the limit walk.** Let T be a tree and $(c(e))$ be conductances on the edges of T such that the associated random walk (X_n) is transient. For every $k \geqslant 0$, the walk visits T_k finitely many times: we can define an infinite path ω^∞ on T by letting $\omega^\infty(k)$ be the last vertex of T_k visited by the walk. Equivalently:

$$\omega^\infty(k) = x \iff x \in T_i \text{ and } \exists n_0, \forall n > n_0 : X_n \in T^x. \tag{1}$$

Let $k \in \mathbb{N}^*$ and $y_0 = o, y_1, y_2, \ldots, y_k$ be k elements of $V(T)$ such that $(y_0, y_1, y_2, \ldots, y_k)$ is a simple path: we can then define

$$\varphi_c(y_0, y_1, y_2, \ldots, y_k) := \mathbb{P}(\omega^\infty(0) = y_0, \omega^\infty(1) = y_1, \ldots, \omega^\infty(k) = y_k). \tag{2}$$

We will refer to this function as the *law of first k-steps of limit walk*. In the case of the biased walk RW_λ, we will denot tho function φ_λ; this will not lead to ambiguities. We finish this section with the following lemma.

Lemma 2.16. *The value of $\varphi_c(y_0, \ldots, y_k)$ depends continuously on any finite collection of the conductances in the network. More precisely, given a finite set $U = \{e_1, \ldots, e_\ell\}$ of edges and a collection $(c(e))$ of conductances, let $\tilde{c}(u_1, \ldots, u_\ell)$ be the family of conductances that coincides with c outside U and takes value u_i at e_i: then the map*

$$\psi_{U,c} : (u_1, \ldots, u_\ell) \mapsto \varphi_{\tilde{c}(u_1, \ldots, u_\ell)}(y_0, \ldots, y_k)$$

is continuous on $(\mathbb{R}_+^*)^\ell$.

3. A few examples

The self-avoiding tree in the plane, which we alluded to in the introduction and will formally introduce in the next section, is sub-periodic but quite inhomogeneous, and the self-avoiding tree in the half-plane sits in none of the classes of trees defined above. To get an intuition of the kind of behavior we should expect or rule out, we gather here a few examples of trees with some atypical features.

3.1. **Trees with discontinuous conductance.** Let $0 < \lambda_0 \leqslant 1$. In the first part of this section, we construct two tree $T, \overline{T}$ with $\lambda_c(T) = \lambda_c(\overline{T}) = \lambda_0$, such that the effective conductances $C(\lambda, T)$ and $C(\lambda, \overline{T})$ of the biased random walk RW_λ on T and $\overline{T}$ satisfy $C(\lambda_c(T), T) = 0$ but $C(\lambda_c(\overline{T}), \overline{T}) > 0$. In the second part, we construct a tree T such that $C(\lambda, T)$ is not continuous on $(\lambda_c, 1)$.

Proposition 3.1. *For every $x \geqslant 1$, there exist two trees $T(x), \overline{T}(x)$ such that:*

- *$br(T(x)) = br(\overline{T}(x)) = x$;*
- *$RW_{1/x}$ is recurrent on $T(x)$ and transient on $\overline{T}(x)$.*

Proof. We will construct spherically symmetric trees satisfying both conditions. Denoting by $[y]$ be the integer part of ym first construct the sequence $(l_i)_{i\in\mathbb{N}^*}$ inductively as follows:

$$l_1 = [x], \quad l_2 = \left[\frac{x^2}{l_1}\right], \quad l_3 = \left[\frac{x^3}{l_1 l_2}\right], \quad \dots, \quad l_n = \left[\frac{x^n}{\prod_{i=1}^{n-1} l_i}\right], \quad \dots$$

and let $T(x)$ be the tree where vertices at distance i from o have l_i children, so that the sizes of the levels of $T(x)$ are given by $|T_n| = \prod_{i=1}^n l_i$. We construct the tree $\overline{T(x)}$ from the degree sequence $(l'_i)_{i\in\mathbb{N}}$ by posing $l'_i = 2l_i$ if i can be written under the form $i = k^2$, and $l'_i = l_i$ otherwise. Notice that $|\overline{T}_n| = 2^{[\sqrt{n}]}|T_n|$.

We first show that both trees have branching number x. Since they are spherically symmetric, it is enough to check that their growth rate is x; the case $x = 1$ is trivial, so assume $x > 1$. From the definition,

$$x^n - \prod_{i=1}^{n-1} l_i \leqslant \prod_{i=1}^{n} l_i \leqslant x^n \quad \text{hence} \quad x^n - x^{n-1} \leqslant |T_n| \leqslant x^n$$

so $gr(T) = x$; the case of $\overline{T}$ follows directly.

The recurrence or transience of the critical random walks can be determined using lemma 2.12:

$$\sum \frac{1}{\lambda_c^n |T_n|} \geqslant \sum \frac{1}{\lambda_c^n x^n} = +\infty$$

so the critical walk on $T(x)$ is recurrent, while for $x > 1$,

$$\sum \frac{1}{\lambda_c^n |\overline{T}_n|} \leqslant \sum \frac{1}{\lambda_c^n (x^n - x^{n-1}) 2^{[\sqrt{n}]}} = \frac{x}{x-1} \sum \frac{1}{2^{[\sqrt{n}]}} < \infty$$

so the critical walk on $\overline{T}(x)$ is transient. In the case $x = 1$ one gets $\sum 2^{-[\sqrt{n}]} < \infty$ instead, and the conclusion is the same. $\square$

Proposition 3.2. *For every $k \in \mathbb{N}^*$ and $\lambda_c \in (0, 1)$, there exists a tree T with critical drift $\lambda_c(T) = \lambda_c$ and such that the ratio $C(\lambda)/(\lambda - \lambda_c)^k$ remains bounded and away from 0 as $\lambda \to \lambda_c^+$.*

Proof. We construct a spherically symmetric tree T which satisfies the conditions of this proposition in a similar way as before. Letting $x = 1/\lambda_c > 1$, define inductively

$$l_1 = [x], \quad l_2 = \left[\frac{x^2}{2^k l_1}\right], \quad l_3 = \left[\frac{x^3}{3^k l_1 l_2}\right], \quad \dots, \quad l_n = \left[\frac{x^n}{n^k \prod_{i=1}^{n-1} l_i}\right], \quad \dots$$

and let T be the spherically symmetric tree with degree sequence (l_i). It is easy to check that $br(T) = x$ like in the previous proposition; in a similar way,

$$x^n - n^k \prod_{i=1}^{n-1} l_i \leqslant n^k \prod_{i=1}^{n} l_i \leqslant x^n \quad \text{hence} \quad \frac{x^n}{n^k} - \frac{x^{n-1}}{(n-1)^k} \leqslant |T_n| \leqslant \frac{x^n}{n^k}.$$

Using Proposition 2.11, the equivalent resistance at parameter $\lambda > \lambda_c$ is given by

$$R(\lambda) = \sum \frac{1}{\lambda^n |T_n|} \geqslant \sum \frac{n^k}{(\lambda x)^n} \sim \frac{C_k}{(\lambda - \lambda_c)^{k+1}}$$

with a lower bound of the same order but with a different constant, leading to the conclusion. $\square$

We end this subsection with the following proposition, showing that discontinuities can occur elsewhere than at λ_c:

Proposition 3.3. *There exists a tree T such that the function $C(\lambda, T)$ is not continuous on $(\lambda_c, 1)$.*

Proof. Let $0 < \lambda_1 < \lambda_2 < 1$. By proposition 3.1, there exists T_a, T_b such that $\lambda_c(T_a) = \lambda_1, \lambda_c(T_b) = \lambda_2$ andq

$$C(\lambda_1, T_a) = 0, C(\lambda_2, T_b) > 0.$$

We construct a tree T as follows $T_1 = \{x_1, x_2\}$ and $T^{x_1} = T_a, T^{x_2} = T_b$, then

$$\lambda_c(T) = \lambda_1.$$

We can see that the function $C(\lambda, T)$ is discontinuous at λ_2. $\square$

Note that continuity properties at $\lambda \geqslant 1$ are actually easier to obtain, and we will investigate them further below.

3.2. **The convergence of law of first k-steps.** If $\lim_{\lambda \to \lambda_c, \lambda > \lambda_c} C(\lambda, T) > 0$, by lemma 6.16 the limit of the function $\varphi^{\lambda,k}(y_1, \dots, y_k)$ when λ decreases to λ_c exists. If $\lim_{\lambda \to \lambda_c, \lambda > \lambda_c} C(\lambda, T) = 0$, the situation is more delicate and we cannot yet conclude on the limit of the function $\varphi^{\lambda,k}(y_1, \dots, y_k)$ when λ decreases to λ_c. Indeed, convergence does not always hold, as we will see in a counterexample. The idea of what follows is easy to describe: we are going to construct a very inhomogeneous tree with various subtrees of higher and higher branching numbers, at locations alternating between two halves of the whole tree; a biased random walk will wander until it finds the first such sub-tree inside which it is transient, and escape to infinity within this subtree with high probability.

Proposition 3.4. *There exists a tree T such that the function $\varphi^{\lambda,1}(y_0, y_1)$ does not converge as $\lambda \to \lambda_c$.*

Notation. Let T, T' be two trees and $A \subset V(T)$. We can construct a new tree by grafting a copy of T' at all the vertices of A; we will denote this new tree by $T \overset{A}{\bigoplus} T'$. Note that for all $x \in A$, $(T \overset{A}{\bigoplus} T')^x \simeq T'$. In the case $A = \{x\}$, we will use the simpler notation $T \overset{x}{\bigoplus} T'$ for $T \overset{\{x\}}{\bigoplus} T'$.

Proof. Fix $\varepsilon > 0$ small enough. By Proposition 3.1, for all $0 < a \leqslant 1$, there exists a tree, denoted by T_a, such that its branching number is $\frac{1}{a}$ and $C(a, T_a) = 0$. Let $H = \mathbb{Z}$, seen as a tree rooted at 0, so that the integers is the vertices of H (see the Figure 1). We are going to construct a tree inductively.

Let $(a_i)_{i \geqslant 1}$ be a decreasing sequence such that $a_1 < 1$ and denoted $a_c := \lim a_i$; assume $a_c > 0$. Choose a sequence b_i such that $b_i \in (a_{i+1}, a_i)$ for all i. First, set $H^0 := (H \overset{-2}{\bigoplus} T_{a_1}) \overset{2}{\bigoplus} T_{a_2}$. We consider the biased random walk RW_{b_1}, then it is recurrent on T_{a_1} and transient on T_{a_2}. On H^0, the biased random walk RW_{b_1} is transient, and in addition we know that it stays eventually within the copy of T_{a_2}. There exists then $N_1 > 2$ such that the probability that the limit walk remains in that copy after time $N_1 - 1$ is greater than $1 - \varepsilon$.

Then we set $H^1 = (H^0 \overset{-N_1}{\bigoplus} T_{a_3})$. On H^1, the walk of bias b_1 is still transient and still has probability at least $1 - \varepsilon$ to escape through the copy of T_{a_2}, because T_{a_3} is grafter too far to be relevant. On the other hand, consider the biased random walk RW_{b_2}: it is still transient on H^1 but only through the new copy of T_{a_3}. There exists then $N_2 > 2$ such that the probability that the limit walk remains in that copy after time $N_2 - 1$ is greater than $1 - \varepsilon$.

We can set $H^2 := (H^1 \overset{N_2}{\bigoplus} T_{a_4})$ and continue this procedure to graft all the trees T_{a_i}, further and further from the origin and alternatively on the left and on the right; we denote by H^∞ the union of all the H^k.

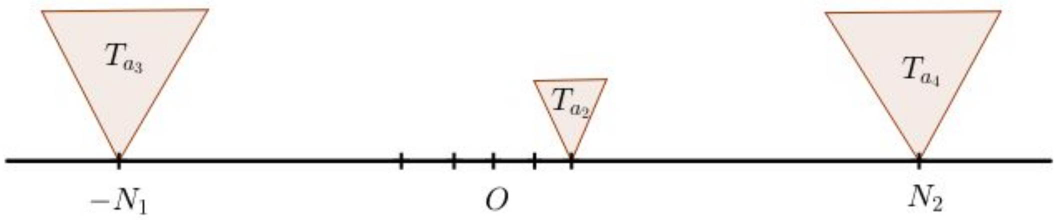

FIGURE 1. Tree H^∞

It remains to show that the function $\varphi^{\lambda,1}(0,1)$ for the biased random walk on the tree H^∞ does not converge. We have $br(H^\infty) = \max_i br(T_{a_i}) = \frac{1}{a_c}$ and $\varphi^{b_i,1}(0,1) \geqslant 1-\varepsilon$ if i is odd while $\varphi^{b_i,1}(0,-1) \geqslant 1-\varepsilon$ if i is even. Then,

$$\forall k \geqslant 0, \begin{cases} \varphi^{b_i,1}(0,1) \geqslant 1-\varepsilon & \text{if } i = 2k+1 \\ \varphi^{b_i,1}(0,1) \leqslant \varepsilon & \text{if } i = 2k+2 \end{cases}$$

This implies that the function $\varphi^{\lambda,1}(0,1)$ does not converge when λ go to a_c. □

The tree we just contructed is taylored to be extremely inhomogeneous. At the other end of the spectrum, some trees have enough structure for all the functions we are considering to be essentially explicit:

Definition 3.5. A tree T is called *periodic* (or *finite type*) if, for all $v \in V(T) \setminus \{o\}$, there is an adjacency-preserving bijection $f : T^v \to T^{f(v)}$ with $f(v)$ in a fixed, finite neighborhood of the root ot T.

Definition 3.6. Let T be a finite tree and $L(T)$ be the set of leafs of T. We set $T^1 = T \overset{L(T)}{\bigoplus} T$, $T^2 = T^1 \overset{L(T^1)}{\bigoplus} T, \dots, T^n = T^{n-1} \overset{L(T^{n-1})}{\bigoplus} T$ for every $n \geqslant 2$. We continue this procedure an infinite number of times to obtain an infinite tree. This infinite tree is called T*-finite type* and denoted by $T^{\infty,T}$. Note that $T^{\infty,T}$ is also a periodic tree.

Definition 3.7. Let T be a periodic tree and $u, v \in V(T)$. We say that u and v have *the same type* if there is an adjacency-preserving bijection $f : T^u \to T^v$. We denote by type (u, v) := $\{w \in V(T^v) :$ w has the same type with u$\}$.

Fact 3.8 (see Lyons [12], theorem 5.1)**.** *Let T be a periodic tree and $(y_0 = o, y_1, y_2, \dots, y_k)$ be a simple path on T. Then the function $\varphi^{\lambda,k}(y_0, y_1, \dots, y_k)$ converges when λ decreases to $\lambda_c(T)$.*

Moreover, the limit of this function is:

$$\lim_{\lambda\to\lambda_c}\varphi^{\lambda,k}(y_0,y_1,\dots,y_k)=\prod_{i=0}^{k-1}\sum_{v\in\ \mathit{type}\ (y_i,y_{i+1})}\lambda_c^{|v|}.$$

To keep this paper self-contained, in the rest of this section we provide a proof of a particular case of fact 3.8:

Proposition 3.9. *Let T be a finite tree and $(y_0=o,y_1,y_2,\dots,y_k)$ be a simple path on $T^{\infty,T}$. Then the function $\varphi^{\lambda,k}(y_0,y_1,\dots,y_k)$ of $T^{\infty,T}$ converges when λ decreases to $\lambda_c(T^{\infty,T})$.*

Before showing the proposition 3.9, we study an explicit example of a tree of finite type.

Example 3.10. We define a finite tree T as follows:

$$\begin{cases} T_1=\{x_1,x_2\}\\ T_2=\{y\}\ \text{ and } x_2 \text{ is parent of } y\\ T_n=\varnothing \text{ for all } n\geqslant 3\end{cases}$$

We can see that $T^{\infty,T}$ is 1-super-periodic. By theorem 2.7, we obtain $br(T^{\infty,T})=gr(T^{\infty,T})$. Recall that l_n is the number of children of a vertex at distance n from o. It is easy to see that $l_{n+1}=l_n+l_{n-1}$ for all $n\geqslant 1$ et $l_1=2,l_2=3$. Then we obtain,

$$l_n=(1+\frac{2\sqrt{5}}{5})(1+\frac{\sqrt{5}}{2})^n+(1-\frac{2\sqrt{5}}{5})(1-\frac{\sqrt{5}}{2})^n \text{ for all } n\geqslant 1.$$

This implies that $\lambda_c(T^\infty)=\frac{2}{1+\sqrt{5}}$. We have

$$\sum_{n=1}^{+\infty}(\sum_{e^-\in T_n^\infty}\lambda_c^n)^{-1}=\sum_{n=1}^{+\infty}\frac{(\frac{1+\sqrt{5}}{2})^n}{l_n}=+\infty.$$

By theorem 2.14, the biased random walk $RW_{\lambda_c(T^{\infty,T})}$ on T^∞ is recurrent. It remains to show that $\varphi^{\lambda,1}(o,x_1)$ converges; after the calculations, we obtain

$$\varphi^{\lambda,1}(o,x_1)=\frac{\lambda}{1+2\lambda\widetilde{C}(\lambda,T^{\infty,T})},$$

and then $\varphi^{\lambda,1}(o,x_1)$ converges to λ_c when λ go to λ_c.

Lemma 3.11. *Let T be a tree such that $\deg o=d_0$ and*

$$\begin{cases} T_1=\{x_1,x_2,\dots,x_{d_0}\}\\ \textit{for all } t\in\{1,2,\dots,d_0\},\lambda_c(T)=\lambda_c(T^{x_i})=\lambda_c \textit{ and } C(\lambda_c,T)=C(\lambda_c,T^{x_i})=0\end{cases}$$

Then for all i, $\widetilde{C}(\lambda,T^o,x_i)=\frac{(d_{x_i}-1)\lambda\widetilde{C}(\lambda,T^{x_i})}{d_0(1+(d_{x_i}-1)\lambda\widetilde{C}(\lambda,T^{x_i}))}$, where $d_{x_i}=\deg x_i$. In particular, if $\frac{\widetilde{C}(\lambda,T^0,x_i)}{\widetilde{C}(\lambda,T^{x_i})}$ converges towards a limit when λ go to λ_c, then this limit is equal to $\frac{(d_{x_i}-1)\lambda_c}{d_0}$.

Proof. We can see that

$$\widetilde{C}(\lambda,T^o,x_i)=\frac{1}{d_0}\left[mc+m^2(1-c)c+m^3(1-c)^2c+\cdots\right]=\frac{1}{d_0}mc\sum_{k=0}^{\infty}(m(1-c))^k,$$

where $m=\frac{(d_{x_i}-1)\lambda}{1+(d_{x_i}-1)\lambda}$ and $c=\widetilde{C}(\lambda,T^{x_i})$. Then we obtain $\widetilde{C}(\lambda,T^o,x_i)=\frac{(d_{x_i}-1)\lambda\widetilde{C}(\lambda,T^{x_i})}{d_0(1+(d_{x_i}-1)\lambda\widetilde{C}(\lambda,T^{x_i}))}$.

If $\frac{\widetilde{C}(\lambda,T^0,x_i)}{\widetilde{C}(\lambda,T^{x_i})}$ converges and as $\widetilde{C}(\lambda_c,T^{x_i})=0$, then $\frac{\widetilde{C}(\lambda,T^0,x_i)}{\widetilde{C}(\lambda,T^{x_i})}$ converges towards $\frac{(d_{x_i}-1)\lambda_c}{d_0}$. □

Proof of proposition 3.9. First, the biased random walk RW_{λ_c} on T^∞ is recurrent (see [12], theorem 5.1). Recall that $L(T)$ is the set of all leafs of finite tree T and S^i be the set of all finite paths starting at origin, ending at one element of $L(T)$ and pass through x_i. We have, for all $x \in L(T), (T^\infty)^x \simeq T^\infty$ and we apply several times successive Lemma 3.11 to obtain

$$\widetilde{C}(\lambda, T^\infty, x_i) = \sum_{\gamma \in S^i} f_1^\gamma(\lambda) f_2^\gamma(\lambda) \cdots f_{|\gamma|}^\gamma(\lambda) \widetilde{C}(\lambda, T^{\gamma_{|\gamma|}}),$$

where $f_i^\gamma(\lambda) = \frac{m_{\gamma_i}\lambda}{m_{\gamma_{i-1}}(1+m_{\gamma_i}\lambda C(\lambda, T^{\gamma_i}))}$ and $m_{\gamma_i} = d_{\gamma_i} - 1$. Since $\widetilde{C}(\lambda, T^{\gamma_{|\gamma|}}) = \widetilde{C}(\lambda, T^\infty)$, then

$$\widetilde{C}(\lambda, T^\infty, x_i) = \sum_{\gamma \in S^i} f_1^\gamma(\lambda) f_2^\gamma(\lambda) \cdots f_{|\gamma|}^\gamma(\lambda) \widetilde{C}(\lambda, T^\infty).$$

By Lemma 6.16, we obtain

$$\varphi^{\lambda,1}(x_i) = \frac{\widetilde{C}(\lambda, T^\infty, x_i)}{\widetilde{C}(\lambda, T^\infty)} = \sum_{\gamma \in S^i} f_1^\gamma(\lambda) f_2^\gamma(\lambda) \cdots f_{|\gamma|}^\gamma(\lambda).$$

We observe that for all $\gamma \in S^i, m_{\gamma_0} = m(\gamma_{|\gamma|})$, this implies that $\varphi^{\lambda,1}(x_i)$ converges when λ decreases towards $\lambda_c(T^\infty)$ and

$$\lim_{\lambda \to \lambda_c(T^\infty)} \varphi^{\lambda,1}(x_i) = \sum_{\gamma \in S^i} \lambda_c^{|\gamma|}. \tag{3}$$

□

Remark 3.12. *The equation (3) gives us a way to calculate the critical value of RW_λ on T^∞, as the solution of the equation*

$$\sum_{i=1}^{m_o} \sum_{\gamma \in S^i} x^{|\gamma|} = 1.$$

4. The continuity of effective conductance

We end the first half of the paper with a few results on the conductance functions of trees, namely we give a criterion for the continuity of $C(\lambda, T)$ in λ and study the set of conductance functions of spherically symmetric trees of bounded degree.

4.1. Left- and right-continuity.

Lemma 4.1. *Let T be a locally finite tree, then $C(\lambda, T)$ is right continuous on $[0, 1]$.*

Proof. We define $S_n := \inf\{k > 0 : d(o, X_k) = n\}$ where X_n is RW_λ. Then

$$C(\lambda, T) = \pi(o) \lim_{n \to +\infty} \mathbb{P}(S_n < S_o).$$

We set $C(\lambda, n) := \pi(o)\mathbb{P}(S_n < S_0)$. It is easy to see that $C(\lambda, n) \geqslant C(\lambda, n+1)$. Moreover, by theorem 2.9, $C(\lambda, n)$ is a continuous increasing function for each n. It implies that $C(\lambda, T)$ is the decreasing limit of increasing functions. Therefore $C(\lambda, T)$ is right continuous. □

Definition 4.2. Let T be a locally finite tree. For each $x \in T$, we let X_n^x denote the biased random walk on T^x (i.e $X_0^x = x, \forall n > 0 : X_n^x \in T^x$). We say that T is *uniformly transient* if

$$\forall \lambda > \lambda_c, \exists \alpha_\lambda > 0, \forall x \in T, \mathbb{P}(\forall n > 0, X_n^x \neq x) \geqslant \alpha_\lambda.$$

It is called *weakly uniformly transient* if there exists a sequence of finite pairwise disjoint cut-sets Π_n, such that

$$\forall \lambda > \lambda_c, \exists \alpha_\lambda > 0, \forall x \in \bigcup_{k=1}^{+\infty} \Pi_k, \mathbb{P}(\forall n > 0, X_n^x \neq x) \geqslant \alpha_\lambda.$$

We can see that if $\lambda_c(T) = 1$, then T is uniformly transient.

Theorem 4.3. *Let T be a uniformly transient tree. Then $C(\lambda, T)$ is left continuous on $(\lambda_c, 1]$.*

Proof. Fix $\lambda_1 > \lambda_c$, we will prove that $C(\lambda, T)$ is left continuous at λ_1. Choose $\lambda_0 \in (\lambda_c, \lambda_1)$. By theorem 2.9, we can find a constant $\alpha > 0$ (does not depend on $\lambda \in [\lambda_0, \lambda_1]$) such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \forall x \in V(T), \mathbb{P}(\forall n > 0, X_n^x \neq x) \geqslant \alpha.$$

We give a family of conductances $c(e)_{e\in E(T)} \in [0,1]^E$, and Y_n that is the associated random walk. Let $A \subset [0,1]^E$ be a subset of elements of $[0,1]^E$ such that Y_n is transient. Then we define the following function

$$A \ni c(e)_{e\in E} \mapsto \psi(c(e)_{e\in E}) := C_{c(e)_{e\in E}}(o \leftrightarrow \infty).$$

Recall that T_k is the collection of all the vertices at distance k from the root: then we have

$$C(\lambda, T) = \psi(\underbrace{\lambda, \lambda, \dots \lambda}_{|T_1|}, \underbrace{\lambda^2, \lambda^2, \dots \lambda^2}_{|T_2|}, \dots).$$

We will abuse notation until the end of the argument, writing

$$\psi(\lambda_1, \lambda_2^2, \lambda_3^3, \dots) \quad \text{for} \quad \psi(\underbrace{\lambda_1, \lambda_1, \dots \lambda_1}_{|T_1|}, \underbrace{\lambda_2^2, \lambda_2^2, \dots \lambda_2^2}_{|T_2|}, \dots)$$

so that in particular $C(\lambda, T) = \psi(\lambda, \lambda^2, \lambda^3, \dots)$.

Let $\varepsilon > 0$, we choose $L \in \mathbb{N}$ such that $(1-\alpha)^L < \varepsilon$. For $\lambda \in (\lambda_0, \lambda_1)$ we have $|C(\lambda_1, T) - C(\lambda, T)| = \left|\psi(\lambda_1, \lambda_1^2, \lambda_1^3, \dots) - \psi(\lambda, \lambda^2, \lambda^3, \dots)\right|$ and by the triangular inequality, we get

$$|C(\lambda_1, T) - C(\lambda, T)| \leqslant \left|\psi(\lambda_1, \dots, \lambda_1^L, b_1) - \psi(\lambda, \dots, \lambda^L, b_1)\right| + \left|\psi(\lambda, \dots, \lambda^L, b_1) - \psi(\lambda, \dots, \lambda^L, b)\right| \tag{4}$$

where $b := (\lambda^{L+k})_{k\geqslant 1}$ and $b_1 := (\lambda_1^{L+k})_{k\geqslant 1}$.

Let $\lambda' \in [\lambda_0, \lambda_1]$ we denote $S_n^{\lambda'}$ the first hitting point of T_n by the random walk with conductances

$$(\underbrace{\lambda, \lambda, \dots \lambda}_{|T_1|}, \underbrace{\lambda^2, \lambda^2, \dots \lambda^2}_{|T_2|}, \dots, \underbrace{\lambda^L, \lambda^L, \dots \lambda^L}_{|T_L|}, \underbrace{(\lambda')^{L+1}, \dots (\lambda')^{L+1}}_{|T_{L+1}|}, \underbrace{(\lambda')^{L+2}, \dots (\lambda')^{L+2}}_{|T_{L+2}|}), \dots$$

We can see that the law of $S_L^{\lambda_1}$ and the law of S_L^{λ} are identical. When the random walk reaches T_L, it returns to o with a probability strictly smaller than $(1-\alpha)^L$. It implies that

$$\left|\psi(\lambda, \dots, \lambda^L, b_1) - \psi(\lambda, \dots, \lambda^L, b)|\right) \leqslant 2(1-\alpha)^L \leqslant 2\varepsilon. \tag{5}$$

It remains to estimate $\left|\psi(\lambda_1, \dots, \lambda_1^L, b_1) - \psi(\lambda, \dots, \lambda^L, b_1)\right|$. By theorem 2.9, we have

$$\psi(\lambda_1, \dots, \lambda_1^L, b_1) > 0 \text{ and } \psi(\lambda, \dots, \lambda^L, b) > 0.$$

We apply the lemma 2.16 to obtain

$$\exists \delta > 0, \forall \lambda \in [\lambda_0, \lambda_1] \text{ such that } \lambda_1 - \lambda < \delta : \left|\psi(\lambda_1, \dots, \lambda_1^L, b_1) - \psi(\lambda, \dots, \lambda^L, b_1)\right| < \varepsilon. \tag{6}$$

We combine (4), (5) and (6) to get

$$\exists \delta > 0, \forall \lambda \in [\lambda_0, \lambda_1] \text{ such that } \lambda_1 - \lambda < \delta : |C(\lambda_1, T) - C(\lambda, T)| \leqslant 3\varepsilon.$$

This implies that $C(\lambda, T)$ is left continuous at λ_1. □

In the same method as in the proof of theorem 4.3, we can prove the slightly stronger result (the proof of which we omit):

Theorem 4.4. *Let T be a weakly uniformly transient tree: then the equivalent conductance $C(\lambda, T)$ is left continuous on $(\lambda_c, 1]$.*

4.2. **Conductance functions.** Let $\mathbb{S}$ denote the set of spherically symmetric trees. Fix $m \in \mathbb{N}^*$, and we set

$$A_m := \{T \in \mathbb{S}; \forall x \in T, \deg x \leqslant m\} \text{ and}$$
$$\mathbb{F}_m := \left\{f \in C^0([0,1]) : \exists T \in A_m, C(\lambda, T) = f(\lambda)\right\}, \mathbb{F} := \left\{f \in C^0([0,1]) : \exists T, C(\lambda, T) = f(\lambda)\right\}.$$

Definition 4.5. Let T^n be a sequence of trees. We say that T^n converges locally towards T^∞ if $\forall k, \exists n_0, \forall n \geqslant n_0, T^n_{\leqslant k} = T^\infty_{\leqslant k}$, where $T_{\leqslant n}$ is a tree defined by

$$\begin{cases} V(T_{\leqslant n}) := \{x \in T, d(0,x) \leqslant n\} \\ E(T_{\leqslant n}) = E_{|V(T_{\leqslant n}) \times V(T_{\leqslant n})} \end{cases}$$

We are now ready to prove Theorem 1.5. We need the following lemma:

Lemma 4.6. *Let $(f_n)_n$ be a sequence of functions in $\mathbb{F}_m$. Assume that f_n converges uniformly towards f. Then, there exists a function $g \in \mathbb{F}_m$ such that*

$$\forall \lambda, f(\lambda) \leqslant g(\lambda).$$

Proof. Let $(T^n)_n$ be a sequence of elements of A_m such that

$$\forall n, f_n(\lambda) = C(\lambda, T^n).$$

Since the degree of vertices of T^n are bounded by m, we can apply the diagonal extraction argument. After renumbering indices, there exists a subsequence of $(T^n)_n$, denoted also by $(T^n)_n$, converges locally towards some tree, denote by T^∞. As for all $n, T^n \in A_m$, then $T^\infty \in A_m$.

We set $g(\lambda) = C(\lambda, T^\infty)$. It remains to show that

$$\forall \lambda, f(\lambda) \leqslant g(\lambda).$$

Assume that there exists λ_0 such that $f(\lambda_0) > g(\lambda_0)$. We set $c = f(\lambda_0) - g(\lambda_0) > 0$. The sequence $f_n(\lambda_0)$ converges towards $f(\lambda_0)$, thus

$$\exists l_1 > 0, \forall n > l_1, f_n(\lambda_0) > f(\lambda_0) - \frac{c}{4}.$$

Moreover the sequence $C_n(\lambda_0, T^\infty) := \pi(o)\mathbb{P}_{\lambda_0}(S_n < S_o)$ decreases towards $g(\lambda_0)$, It implies that

$$\exists l_2 > 0, \forall n > l_2, C_n(\lambda_0, T^\infty) < g(\lambda_0) + \frac{c}{4}.$$

We take $l > 0$ such that

$$\begin{cases} f_l(\lambda_0) > f(\lambda_0) - \frac{c}{4} \\ C_l(\lambda_0, T^\infty) < g(\lambda_0) + \frac{c}{4} \end{cases}$$

We have $C_l(\lambda_0, T^l) = C_l(\lambda_0, T^\infty)$, then $C_l(\lambda_0, T^l) < g(\lambda_0) + \frac{c}{4}$.

Moreover, the sequence $C_k(\lambda_0, T^l)$ decreases towards $f_l(\lambda_0)$ when k goes to $+\infty$, thus

$$f(\lambda_0) - \frac{c}{4} < f_l(\lambda_0) < g(\lambda_0) + \frac{c}{4}.$$

Then we obtain $c < \frac{c}{4}$. It is a contradiction. □

Remark 4.7. *The lemma 4.6 is still valid if we take $A_m := \{T \in \mathbb{T}; \forall x \in T, \deg x \leqslant m\}$ and $(f_n)_n$ converges simply to f, where $\mathbb{T}$ denote the set of locally finite trees.*

Proof of theorem 1.5. Fix a diagonal extraction and we take the function $g(\lambda)$ as in the proof of the lemma 4.6. We will prove that $f = g$.

By lemma 4.6, we have $f(\lambda) \leqslant g(\lambda)$. Assume that there exists λ_0 such that $0 < f(\lambda_0) < g(\lambda_0)$. We prove that

$$\forall \lambda < \lambda_0, f(\lambda) = 0.$$

By proposition 2.11, if we set $\beta_0 = \frac{1}{\lambda_0}$, then

$$\begin{cases} \forall n, R(\lambda_0, T^n) = \sum_{k=1}^{+\infty} \frac{\beta_0^k}{|T^n_k|} \\ R(\lambda_0, T^\infty) = \sum_{k=1}^{\infty} \frac{\beta_0^k}{|T^\infty_k|} \end{cases}$$

We write

$$R(\lambda_0, T^n) = \sum_{k=1}^{+\infty} \frac{\beta_0^k}{|T_k^n|} = \sum_{k\leqslant n} \frac{\beta_0^k}{|T_k^n|} + \sum_{k>n} \frac{\beta_0^k}{|T_k^n|}.$$

As $\forall k \geqslant n, |T_k^n| = |T_k^\infty|$, then

$$R(\lambda_0, T^n) = \sum_{k\leqslant n} \frac{\beta_0^k}{|T_k^\infty|} + \sum_{k>n} \frac{\beta_0^k}{|T_k^n|}.$$

We know that

$$\begin{cases} \lim_{n\to\infty} R(\lambda_0, T^n) = \frac{1}{f(\lambda_0)} < \infty \\ \lim_{n\to\infty} R(\lambda_0, T^\infty) = \frac{1}{g(\lambda_0)} < \frac{1}{f(\lambda_0)} \end{cases}$$

We obtain

$$\lim_{n\to+\infty} \sum_{k>n} \frac{\beta_0^k}{|T_k^n|} = \frac{1}{f(\lambda_0)} - \frac{1}{g(\lambda_0)} > 0.$$

Now we take $\beta > \beta_0$ and we apply the proposition 2.11 in order to get

$$R(\frac{1}{\beta}, T^n) = \sum_{k=0}^{+\infty} \frac{\beta^k}{|T_k^n|} > \sum_{k>n} \frac{\beta^k}{|T_k^n|} \geqslant (\frac{\beta}{\beta_0})^n \sum_{k>n} \frac{\beta_0^k}{|T_k^n|} \geqslant (\frac{\beta}{\beta_0})^n (\frac{1}{f(\lambda_0)} - \frac{1}{g(\lambda_0)}).$$

It implies that $\lim_{n\to+\infty} f_n(\frac{1}{\beta}) = \lim_{n\to+\infty} \frac{1}{R(\frac{1}{\beta}, T^n)} = 0$. It means

$$\forall \lambda < \lambda_0, f(\lambda) = 0.$$

As $f \neq 0$, we define $\lambda_p := \inf\{0 \leqslant \lambda \leqslant 1 : f(\lambda) > 0\}$. We proved that

$$\forall \lambda > \lambda_p, f(\lambda) = g(\lambda).$$

As the sequence $(f_n)_n$ converges uniformly to f, then f is continuous, and then $f(\lambda_p) = 0$. By lemma 4.1, g is right continuous. Then we obtain

$$f(\lambda_p) = \lim_{\lambda\to\lambda_p} f(\lambda) = \lim_{\lambda\to\lambda_p} g(\lambda) = g(\lambda_p) = 0.$$

Moreover f and g are two increasing functions, then $\forall \lambda \in [0, 1], f(\lambda) = g(\lambda)$. □

5. Self-avoiding walks

5.1. **Walks and bridges.** In this section, we review some definitions on the self-avoiding walk, bridges and connective constant (see [6],[17]). Denote by c_n the number of self-avoiding walks of length n, starting at origin on the graph considered. If G is transitive, the sequence $c_n^{1/n}$ converges to a constant when n goes to infinity. This constant is called the connective constant of G.

Definition 5.1. An *n-step bridge* in the plane $\mathbb{Z}^2$ (or half-plane $\mathbb{H}$) is an n-step self-avoiding walk (SAW) γ such that

$$\forall i = 1, 2, \dots, n, \quad \gamma_1(0) < \gamma_1(i) \leqslant \gamma_1(n)$$

where $\gamma_1(i)$ is the first coordinate of $\gamma(i)$. An *n-step zero-bridge* is an n-step SAW γ such that $\gamma_1(0) \leqslant \gamma_1(i) \leqslant \gamma_1(n), \forall i = 1, 2, \dots, n$. Let b_n denote the number of all n-step bridges with $\gamma(0) = 0$. By convention, set $b_0 = 1$.

We have $b_{m+n} \geq b_m \cdot b_n$, hence we can define

$$\mu_b = \lim_{n\to+\infty} b^{\frac{1}{n}} = \sup_n b_n^{\frac{1}{n}}.$$

Moreover, $b_n \leq \mu_b^n \leq \mu^n$.

Definition 5.2. An *n-step half-space* walk is an n-step SAW γ with $\gamma_1(0) < \gamma_1(i), \forall i$.

We set h_n is the number of all n-step half-space walk with $\gamma(0) = 0$.

Definition 5.3. The *span* of an n-step SAW γ is

$$\max_{0\leq i\leq n}\gamma_1(i)-\min_{0\leq i\leq n}\gamma_1(i).$$

Definition 5.4. Given a bridge (respectively a zero-bridge) γ of length n, γ is called an *irreducible bridge* (respectively *irreducible zero-bridge*) if it can not be decomposed into two bridges (respectively zero-bridges) of length strictly smaller than n. It means, we can not find $i\in[1,n-1]$ such that $\gamma_{|[0,i]},\gamma_{|[i,n]}$ are two bridges (respectively zero-bridges). The set of all irreducible-bridges is denoted by $iSAW$.

5.2. **Kesten's and Lawler's measures.** For this section, we refer the reader to ([9],[4]) for a more precise description. Denote by SAW_∞ the set of all self-avoiding walks on the plane $\mathbb{Z}^2$ or half-plane $\mathbb{H}$. In this part, we review the Kesten measure. He defined a probability measure on the SAW_∞ of half-plane from the finite bridges. We let $\mathbb{B}$ denote the set of bridges starting at origin and B^0 the set of zero-bridges starting at origin. We let also $\mathbb{I}$ denote the set of irreducible bridges starting at origin and $\mathbb{I}^0$ the set of irreducible zero-bridges starting at origin. Let p_n denote the number of irreducible bridges starting at origin, of length n and q_n denote the number of irreducible zero-bridges starting at origin of length n.

We will define a notion of concatenation of paths. If $\gamma^1=[\gamma^1_0,\gamma^1_1,\dots,\gamma^1_m]$ and $\gamma^2=[\gamma^2_0,\gamma^2_1,\dots,\gamma^2_n]$ are two SAWs, we define $\gamma^1\oplus\gamma^2$ to be the $(m+n)$-step walk (not necessarily self-avoiding walk)

$$\gamma^1\oplus\gamma^2:=\left[0,\gamma^1_1,\dots,\gamma^1_m,\gamma^1_m+\gamma^2_1-\gamma^2_0,\dots,\gamma^1_m+\gamma^2_n-\gamma^2_0\right].$$

Similarly, we can define $\gamma^1\oplus\gamma^2\oplus\cdots\oplus\gamma^k$. We begin with the following equality

Fact 5.5 (Kesten [9], Theorem 5). *We have*

$$\sum_{n=1}^{+\infty}\frac{p_n}{\mu^n}=1.$$

Let us now to define the Kesten measure on the SAW_∞ in the half-plane. We fix $\beta\leqslant\frac{1}{\mu}$ and let $\mathbb{Q}^\beta$ denote the probability measure on $\mathbb{I}$ defined by

$$\mathbb{Q}^\beta(\omega)=\frac{\beta^{|\omega|}}{Z_\beta},\omega\in\mathbb{I}$$

where $Z_\beta=\sum_{\omega\in\mathbb{I}}\beta^{|\omega|}$. By the fact 5.5 and the remark 5.6 below, Z_β is finite and thus $\mathbb{Q}^\beta$ is a probability measure on $\mathbb{I}$.

Remark 5.6. *We have also $\Lambda_\beta<\infty$ if $\beta<\frac{1}{\mu}$. If $\beta>\frac{1}{\mu}$ then $Z_\beta=+\infty$, and then $\mathbb{Q}^\beta$ can not be defined.*

If $k\geqslant 1$, we consider the product space $\mathbb{I}^k$ and define the product probability measure $\mathbb{Q}^\beta_k$. We write $\mathbb{Q}^\beta_j$ for an extension to SAW in $\mathbb{H}$ as follows, $\mathbb{Q}^\beta(\omega)=0$ if ω is not of form $\omega^1\oplus\omega^2\oplus\cdots\oplus\omega^k$ and

$$\mathbb{Q}^\beta_j(\mathbb{H}\setminus\mathbb{I}^k)=0;\mathbb{Q}^\beta_j(\omega^1\oplus\omega^2\oplus\cdots\oplus\omega^k)=\mathbb{Q}^\beta(\omega^1)\times\mathbb{Q}^\beta(\omega^2)\times\cdots\times\mathbb{Q}^\beta(\omega^k).$$

We define $\mathbb{Q}^\beta_\infty$ on $\mathbb{I}^\infty$, it is called the β-Kesten measure on SAW_∞ in half-plane.

Fact 5.7. *Under the β-Kesten measure, the infinite self-avoiding walk, denoted by $\omega^{\infty,\beta}_K$, does not reach the line Ox almost surely. Moreover, if $\beta<\frac{1}{\mu}$, we have then*

$$\mathbb{P}(\limsup_n\Re\omega^{\infty,\beta}_K(n)=+\infty)=1;\mathbb{P}(\liminf_n\Re\omega^{\infty,\beta}_K(n)=-\infty)=1.$$

Now, we define an other probability measure on SAW_∞ in half-plane from the finite zero-bridges. Then

Fact 5.8 (see [4]). *For every $\beta \in (0, \frac{1}{\mu}]$, we have*

$$\sum_{n=0}^{+\infty} q_n \beta^n < +\infty.$$

In the same way, we can define a probability measure $\mathbb{L}^\beta$ on $\mathbb{I}_0$. The infinite half-plane SAW starting at 0 is obtained by choosing

$$\omega^1 \oplus \omega^2 \oplus \dots,$$

where $\omega^1, \omega^2, \dots$ are independent; ω^1 is chosen from $\mathbb{L}^\beta$; ω^2, ω^3 ... are chosen from $\mathbb{Q}^\beta$. The law of infinite half-plane SAW is called β-Lawler measure.

Fact 5.9. *Under the β-Lawler measure, the infinite self-avoiding walk, denoted by $\omega_L^{\infty,\beta}$, reaches the line Ox with a probability p which satisfy $0 < p < 1$ and it reaches the line Ox a finite number of times almost surely. Moreover, if $\beta < \frac{1}{\mu}$, then*

$$\mathbb{P}(\limsup_n \Re\omega_L^{\infty,\beta}(n) = +\infty) = 1 \quad \text{and} \quad \mathbb{P}(\liminf_n \Re\omega_L^{\infty,\beta}(n) = -\infty) = 1.$$

5.3. **The self-avoiding tree.** Consider the self-avoiding walks in the lattice $\mathbb{Z}^2$ starting at the origin. We construct a tree $T_{\mathbb{Z}^2}$ from these self-avoiding walks: the vertices of $T_{\mathbb{Z}^2}$ are the finite self-avoiding walks and two such vertices joined when one path is an extension by one step of the other. Formally, denote by Ω_n the set of self-avoiding walks of length n starting at the origin and $V := \bigcup_{n=0}^{+\infty} \Omega_n$. Two elements $x, y \in V$ are adjacent if one path is an extension by one step of the other. We then define $T_{\mathbb{Z}^2} = (V, E)$. We can define with the same way for $T_{\mathbb{H}}, T_{\mathbb{Q}}$, where $\mathbb{H}$ is a half-plane and $\mathbb{Q}$ is a quarter-plane.

We know that $gr(T_{\mathbb{Z}^2}) = br(T_{\mathbb{Z}^2}) = \mu$ where μ is the connective constant of lattice $\mathbb{Z}^2$. We calculate the branching number and the growth rate of $T_{\mathbb{H}}$ and $T_{\mathbb{Q}}$, that is the contents of the following proposition.

Proposition 5.10. *Let $T_{\mathbb{H}}, T_{\mathbb{Q}}$ be defined as above. Then,*

$$gr(T_{\mathbb{H}}) = br(T_{\mathbb{H}}) = gr(T_{\mathbb{Q}}) = br(T_{\mathbb{Q}}) = \mu,$$

where μ is the connective constant of the lattice $\mathbb{Z}^2$.

Notation. In [9], Kesten proved that all bridges in a half-plane can be decomposed into a sequence of irreducible bridges in a unique way. For every $m \in N^*$, we set:

$$A_m := \{\omega \in iSAB, |\omega| \leqslant m\}.$$

An infinite self-avoiding walk is called "m-good" if it possesses a decomposition into irreducible bridges in A_m. We can construct a tree T^m from these m-good walks, which we will refer to as the *m-good tree.*

Proof of proposition 5.10. We know that (see [1], [7]) there exists a constant B and $n_0 \in \mathbb{N}$ such that $\forall n > n_0 : c_n \leqslant b_n e^{B\sqrt{n}}$. This implies that $gr(T_{\mathbb{H}}) = \mu$. Let b_n^m be the number of bridges of length n which possess a decomposition in A_m. Then, $\forall n, |T_n^m| \geqslant b_n^m$.

We have $T^m \subset T_{\mathbb{H}}$, then $br(T^m) \leqslant br(T_{\mathbb{H}})$. Moreover, T^m is m-super-periodic, so we can apply theorem 2.7 to get $br(T^m) = gr(T^m)$. Then $\forall m, br(T_{\mathbb{H}}) \geqslant br(T^m) = gr(T^m)$. We will prove that $\lim gr(T^m) = \mu$. We have $b_{m+n} \geqslant b_m b_n$, $b_{n_1+n_2}^m \geqslant b_{n_1}^m b_{n_2}^m$ and $\mu = \lim_{n\to\infty} b_n^{\frac{1}{n}}$. Fix $\varepsilon > 0$, there exists n_0 such that

$$\forall n \geqslant n_0, \left|\mu - (b_n)^{\frac{1}{n}}\right| \leqslant \varepsilon.$$

For each $n > n_0$, $b_n = b_n^n$ and $b_n^{kn} \geqslant (b_n^n)^k = (b_n)^k$ by sub-additivity. It implies that

$$(b_{kn}^n)^{\frac{1}{kn}} \geqslant (b_n)^{\frac{1}{n}}.$$

The sequence $(b_l^n)^{\frac{1}{l}}$ increases toward μ_n, then $(b_{kn}^n)^{\frac{1}{kn}} \underset{k\to\infty}{\to} \mu_n \leqslant gr(T^n)$. We obtain $gr(T^n) \geqslant \mu - \varepsilon$ and then $\mu \geqslant gr(T^n) \geqslant \mu - \varepsilon, \forall n \geqslant n_0$. This implies that $\lim gr(T^n) = \mu$ and then $br(T_{\mathbb{H}}) \geqslant \mu$.

We apply the proposition 2.3 in order to get $br(T_{\mathbb{H}}) \leqslant gr(T_{\mathbb{H}}) = \mu$. This implies that $br(T_{\mathbb{H}}) = \mu$ and in the same method we obtain $gr(T_{\mathbb{Q}}) = br(T_{\mathbb{Q}}) = \mu$. □

5.4. **Self-avoiding walks in a wedge.** Let C_θ be a cone of angle θ of $\mathbb{H}$. Denote by T_θ the tree which is constructed from the self-avoiding walks in C_θ.

Theorem 5.11. *Denote by $br(T_\theta)$ (respectively $gr(T_\theta)$) the branching number (respectively the growth rate) of T_θ. Then $br(T_\theta) = gr(T_\theta) = \mu$ where μ is the connective constant of $\mathbb{Z}^2$.*

In order to prove this theorem, we have to show the convergence of the connective constants of strips of $\mathbb{H}$ with increasing widths and the associated convergence of the branching numbers of the trees constructed from self-avoiding walks of these strips.

The convergence of connective constant. Let $(B_L)_L$ be the sequence of strips of $\mathbb{H}$ where B_L is a strip between three lines $\Im z = 0$ and $\Im z = L$. We show the convergence of connective constant of B_L towards the connective constant of $\mathbb{Z}^2$. We need the following lemma.

Lemma 5.12 (The subadditivity property)**.** *Let L, n be two positive natural numbers, we denote by b_n^L the number of bridges of length n starting at origin in the strip B_L. Then,*

$$\begin{cases} \forall L, n, m \in \mathbb{N}^* : b_{n+m}^{2L} \geqslant b_m^L b_n^L \\ \forall L, n, k \in \mathbb{N}^* : b_{kn}^{2L} \geqslant (b_n^L)^k \end{cases}$$

Remark 5.13. *$b_{kn}^{kL} \geqslant (b_n^L)^k$ is much easier by sub-additivity.*

Proof of Lemma 5.12. We divide the strip B_{2L} into two small strip of size L, B_{2L}^1, B_{2L}^2 (see the figure 1)

We denote by S_z the symmetry with respect to the line which goes through z and orthogonal to the line Ox. We consider γ_1, γ_2 two bridges in the strip B_{2L}^1 of length m and n, we concatenate γ_1, γ_2, we obtain a path $\gamma_{12} := \gamma_1 \oplus \gamma_2$.

We can see thatγ_{12} is a bridge of B_{2L} of length $m + n$. This implies that

$$\text{for all } L, n, m \in \mathbb{N}^*, b_{n+m}^{2L} \geqslant b_m^L b_n^L.$$

Next, if one takes the third bridge γ_3 of B_{2L}^1 of length p, we concatenate $\gamma_1, \gamma_2, \gamma_3$ as follows.

$$\begin{cases} \gamma_{123} = \gamma_{12} \oplus \gamma_3 \text{ if } \Re(\gamma_{12}(|\gamma_{12}|)) \leqslant L \\ \gamma_{123} = \gamma_{12} \oplus S_{\gamma_{12}(|\gamma_{12}|)}(\gamma_3) \text{ if } \Re(\gamma_{12}(|\gamma_{12}|)) > L \end{cases}$$

We can see that γ_{123} is a bridge of B_{2L} of length $m + n + p$. If we take $m = n = p$, then $b_{3n}^{2L} \geqslant (b_n^L)^3$. We repeat the same strategy to obtain the result of lemma 5.12. □

Proposition 5.14. *We denote by μ_L the connective constant of the strip B_L, then $\lim_L \mu_L = \mu$ where μ is the connective constant of $\mathbb{Z}^2$.*

Proof. We define $b_n^{\mathbb{Q}}$ the number of bridges of $\mathbb{Q}$ of length n, then

$$\begin{cases} \forall L : \lim_n (b_n^L)^{\frac{1}{n}} = \mu_L \\ \forall L : b_L^L = b_L^{\mathbb{Q}} \\ \lim_n (b_n^{\mathbb{Q}})^{\frac{1}{n}} = \mu \\ \forall L, n, k : b_{kn}^{2L} \geqslant (b_n^L)^k \end{cases}$$

From these relations, we prove that $\lim_L \mu_L = \mu$. Let $\varepsilon > 0$, we take n_0 such that: $\left|(b_n^{\mathbb{Q}})^{\frac{1}{n}} - \mu\right| \leqslant \varepsilon, \forall n \geqslant n_0$. The sequence $(b_{kn_0}^{2n_0})^{\frac{1}{kn_0}}$ converges towards μ_{2n_0}. Moreover $b_{kn_0}^{2n_0} \geqslant (b_{n_0}^{n_0})^k$, thus $(b_{kn_0}^{2n_0})^{\frac{1}{kn_0}} \geqslant (b_{n_0}^{n_0})^{\frac{1}{n_0}}$.

By making k tend to $+\infty$, we obtain $\mu_{2n_0} \geqslant \mu - \varepsilon$. As the sequence $(\mu_L)_L$ is an increasing sequence, thus $\forall L > 2n_0 : \mu_L > \mu - \varepsilon$.

This implies that the sequence μ_L converge towards μ when L goes to $+\infty$. □

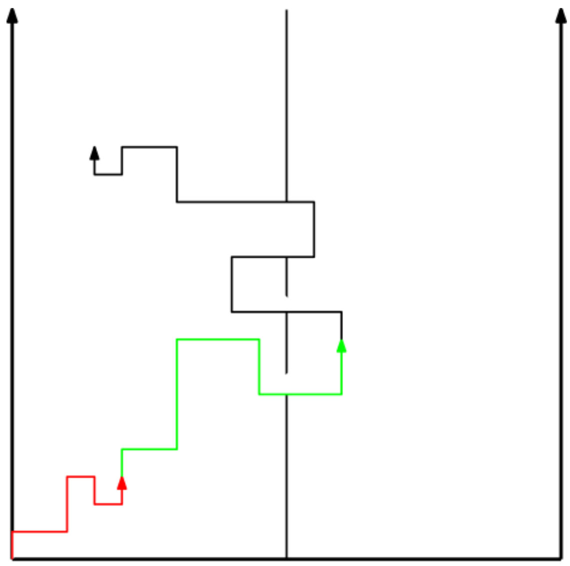

FIGURE 2. A concatenation of 3 bridges in B^1_{2L}.

Proposition 5.15. $\lim_L br(T_{B_L}) = \mu$.

Proof. Recall the definition of A_m in the proof of Proposition 5.10. An infinite self-avoiding walk of B_L is called "m-good walk" if it possesses a decomposition into irreducible bridges in A_m. We construct a tree $T^m_{B_L}$ from these m-good walks. We set $b^{L,m}_n$ be the number of bridges of B_L of length n which possess a decomposition in A_m. We know that all bridges in a half-plane can decompose into a sequence of irreducible bridges of the unique way. This implies that all bridges in the strip B_L can decompose into a sequence of irreducible bridges of the unique way. Then for all n, $\left|(T^m_{B_L})_n\right| \geqslant b^{L,m}_n$.

We obtain, $\begin{cases} \forall L, n, k : b^{2L}_{nk} \geqslant (b^L_n)^k \\ \forall L, m, n, k : b^{2L,m}_{nk} \geqslant (b^{L,m}_n)^k \end{cases}$

Let $\varepsilon > 0$, we apply the proposition 5.14, there exists L_0 such that: $\mu \geqslant \mu_{L_0} > \mu - \varepsilon$.

Moreover, $\mu_{L_0} = \lim (b_n^{L_0})^{\frac{1}{n}}$, there is also m: $(b_m^{L_0})^{\frac{1}{m}} > \mu_{L_0} - \varepsilon$. Then,

$$(b_{km}^{2L_0,m})^{\frac{1}{km}} \geqslant (b_m^{L_0,m})^{\frac{1}{m}} = (b_m^{L_0})^{\frac{1}{m}} \geqslant \mu_{L_0} - \varepsilon \geqslant \mu - 2\varepsilon.$$

We obtain $\overline{gr(T^m_{B_{2L_0}})} \geqslant \mu - 2\varepsilon$. As $T^m_{B_{2L_0}}$ is $(m+2L_0)$-super-periodic and $\overline{gr(T^m_{B_{2L_0}})} < +\infty$, we apply the theorem 2.7 to get $gr(T^m_{B_{2L_0}})$ exists and $gr(T^m_{B_{2L_0}}) = br(T^m_{B_{2L_0}})$. Moreover $T^m_{B_{2L_0}} \subset T_{B_{2L_0}}$, then $br(T_{B_{2L_0}}) \geqslant \mu - 2\varepsilon$. The sequence $br(T_{B_L})$ is an increasing sequence, we obtain:

$$\forall L \geqslant 2L_0 : br(T_{B_L}) \geqslant \mu - 2\varepsilon.$$

Moreover $br(T_{B_L}) \leqslant \mu$ because $T_{B_L} \subset T_{\mathbb{H}}$. This implies that $\lim_L br(T_{B_L}) = \mu$. □

Proof of Theorem 5.11. Let $\varepsilon > 0$, we apply the proposition 5.14 and 5.15, there exists a constant L such that: $br(T_{B_L}) > \mu - \varepsilon$ and $gr(T_{B_L}) > \mu - \varepsilon$.

We take N such that: $N \tan\theta > L$. We have the cone C_θ contains a strip of size greater than L, origin at $(N\tan\theta, 0)$. This implies that $br(T_\theta) > \mu - \varepsilon; gr(T_\theta) > \mu - \varepsilon$.

As ε is arbitrary, then $br(T_\theta) = \mu$ and $gr(T_\theta) = \mu$. □

5.5. **Continuity of C on $T_{\mathbb{H}}$.** Now, we apply the results in Section 4.1 for the self-avoiding trees $T_{\mathbb{H}}$ and $T_{\mathbb{Z}^2}$.

Theorem 5.16. *The function $C(\lambda, T_{\mathbb{H}})$ (or $C(\lambda, T_{\mathbb{Z}^2})$) is continuous on $(\frac{1}{\mu}, 1)$, where μ is the connective constant of the lattice $\mathbb{Z}^2$.*

Proof. The right continuity of $C(\lambda, T_{\mathbb{H}})$ is a consequence of the lemma 4.1.

In order to prove the left continuous, we seek to apply the theorem 4.4. For this, we prove that $T_{\mathbb{H}}$ is weakly uniformly transient. In the half-plane $\mathbb{H}$, we define a sequence of rectangles $(R_n)_{n\geqslant 1}$ where R_n is the rectangle with 4 vertices

$$(-n,0); (-n,n); (n,n); (n,0).$$

We define a sequence of pairwise disjoint cut-sets from these rectangles as follows:

$$\Pi_n := \left\{ \gamma : \gamma(|\gamma|) \in R_n \text{ and } \forall k < |\gamma|, \gamma(k) \in \overset{o}{R}_n \right\}.$$

We set $\Gamma := \bigcup \Pi_n$. It remains to verify that

$$\forall \lambda > \lambda_c (= \frac{1}{\mu}), \exists \alpha_\lambda > 0, \forall x \in \Gamma, \mathbb{P}(\forall n > 0, X_n^x \neq x) \geqslant \alpha_\lambda.$$

Recall that $T_{\mathbb{Q}}$ denote the self-avoiding tree from a quarter-plane. We can see that, for every $x \in \Gamma$, T^x contains the tree $T_{\mathbb{H}}$ or $T_{\mathbb{Q}}$. We conclude thank to the proposition 5.10. □

Remark 5.17. *The self-avoiding trees $T_{\mathbb{H}}$ and $T_{\mathbb{Z}^2}$ are not uniformly transient.*

Recall that B_n is a strip of $\mathbb{H}$ and T_{B_n} is the self-avoiding tree which is constructed from self-avoiding walks in B_n. Let $f_n(\lambda) := C(\lambda, T_{B_n})$.

Theorem 5.18. *The sequence of functions $(f_n)_n$ converges uniformly towards a continuous function f if and only if RW_{λ_c} on $T_{\mathbb{H}}$ is recurrent, where $\frac{1}{\lambda_c} = \mu$ is the connective constant of lattice $\mathbb{Z}^2$.*

In order to prove the theorem 5.18, we need the following lemma.

Lemma 5.19. *For all k, $f_k(\frac{1}{\mu}) = 0$.*

Proof. We use the theorem 2.14 for this proof.

We fix $k \in \mathbb{N}^*$ and for each n, we define the rectangle R_n with 4 vertices

$$(-n,0); (-n,k); (n,k); (n,0).$$

We define a sequence of cut-sets from these rectangles as follows

$$\Pi_n := \{\gamma : \Re(\gamma(|\gamma|)) \in \{-n, n\}; \forall k < |n|, |\Re(\gamma(k)| < n \text{ and } 0 \leqslant \Im(\gamma(k)) \leqslant k\}.$$

We will estimate $\left(\sum_{e\in\Pi_n} c(e)\right)^{-1}$. We can see that a self-avoiding walk $\gamma \in \Pi_n$ of length i can expand into a bridge of length $i+k+1$ and as $\gamma \in \Pi_n$ then $n \leqslant |\gamma| \leqslant 2kn$, it implies that

$$\sum_{e\in\Pi_n} c(e) \leqslant \sum_{i=n}^{2kn} \frac{b_{i+k+1}}{\mu^i} \leqslant 3^k \sum_{i=n}^{2kn} \frac{b_i}{\mu^i} \leqslant 2k3^k n,$$

and then

$$\sum_n \left(\sum_{e\in\Pi_n} c(e)\right)^{-1} \geqslant \sum_n \frac{1}{2k3^k n} = +\infty.$$

It means $RW_{\frac{1}{\mu}}$ on T_{B_k} is recurrent and then $f_k(\frac{1}{\mu}) = 0$. □

Proof of Theorem 5.18. Assume that $(f_n)_n$ converges towards a continuous function f. We will prove that $RW(\frac{1}{\mu}, T_{\mathbb{H}})$ is recurrent.

We set $g(\lambda) = C(\lambda, T_{\mathbb{H}})$, we prove that $\forall \lambda > \frac{1}{\mu}, f(\lambda) = g(\lambda)$.

We fix $\lambda > \frac{1}{\mu}$ and define a sequence of pairwise disjoint cut-sets O_n by considering the first time reaches the rectangles (as in the proof of the theorem 5.16).

Let T be an arbitrary tree and C denote its cut set. We set $T(0 \leftrightarrow C) := \{x \in T : \exists y \in C, x \leqslant y\}$. We can see that

$$\forall n, T_{B_n}(0 \leftrightarrow O_n) = T_{\mathbb{H}}(0 \leftrightarrow O_n).$$

We proved that $\lim \lambda_c(T_{B_n}) = \frac{1}{\mu}$, then we can find $l > 0$ such that $\lambda_c(T_{B_l}) < \lambda$. Then we set

$$m = C(\lambda, T_{B_l}) > 0.$$

Then, there exists $k > 0$ such that

$$\forall n > k, \begin{cases} \mathbb{P}_\lambda(0 \leftrightarrow O_n) > g(\lambda) - \varepsilon \\ (1-m)^n < \varepsilon \end{cases}$$

We obtain

$$\forall i > k + l, f_i(\lambda) > g(\lambda) - 2\varepsilon$$

Then $g(\lambda) \geqslant f(\lambda) > g(\lambda) - 2\varepsilon$. This implies that $f(\lambda) = g(\lambda)$.

Since f is a continuous function, thus

$$f(\frac{1}{\mu}) = \lim_{\lambda \to \frac{1}{\mu}} f(\lambda) = \lim_{\lambda \to \frac{1}{\mu}} g(\lambda) = g(\frac{1}{\mu}).$$

By lemma 5.19, we have $f(\frac{1}{\mu})$, and then $RW_{\frac{1}{\mu}}$ on $T_{\mathbb{H}}$ is recurrent.

Conversely, if $RW_{\frac{1}{\mu}}$ on $T_{\mathbb{H}}$ is recurrent, it is easy to see that

$$\forall \lambda \in [0, 1], f(\lambda) = g(\lambda).$$

By theorem 5.16 and moreover $g(\frac{1}{\mu}) = 0$, we have then g is continuous function and then f is continuous. □

In the same way, we can prove the following:

Proposition 5.20. *With the same notations as in the proof of the proposition 5.10, set $f_n(\lambda) := C(\lambda, T^n)$. Then the sequence of functions $(f_n)_n$ converges towards a continuous function f if and only if RW_{λ_c} on $T_{\mathbb{H}}$ is recurrent, where $\frac{1}{\lambda_c} = \mu$ is the connective constant of lattice $\mathbb{Z}^2$.*

6. The biased walk on the self-avoiding tree

We now begin the study of our main object of interest, which is the biased random walk on the self-avoiding tree. We will use the results obtained in the previous section to prove properties of the limit walk; in the next section, we will gather a few natural conjectures.

6.1. **The limit walk.** Let $\lambda \in [0, +\infty]$. We consider the biased random walk RW_λ on $T_{\mathbb{H}}$ or $T_{\mathbb{Z}^2}$. By proposition 5.10, we have $\lambda_c(T_{\mathbb{H}}) = \lambda_c(T_{\mathbb{Z}^2}) = \frac{1}{\mu}$. We take $\lambda > \lambda_c$, the biased random walk is transient. Then almost surely, the random walk do not reach T_k after n steps, with n large enough. We can then define the limit walk, denoted by ω_λ^∞ in the following way:

$$\omega_\lambda^\infty(i) = x_i \Leftrightarrow \begin{Bmatrix} x_i \in T_i \\ \exists n_0, \forall n > n_0 : X_n \in T^{x_i} \end{Bmatrix}$$

We can see that ω_λ^∞ is a random ray. Denote by $\mathbb{P}_\lambda$ be the law of ω_λ^∞ in the half-plane and $\mathbb{Q}_\lambda$ be the law of ω_λ^∞ in the plane. We can see also that $\mathbb{P}_\lambda$ (respectively $\mathbb{Q}_\lambda$) is a probability measure on SAW_∞ in the half-plane (respectively the plan).

We remove all of finite branch of T_R where R is a regular lattice, then we obtain a tree which have no leaf, denoted by TSL_R.

6.2. **The case $\lambda = +\infty$ and percolation.** First, we review some definitions on the percolation theory. Percolation was introduced by Broadbent and Hammersley in 1957 [3]. For $p \in [0, 1]$, we consider the triangular lattice $\mathbb{T}$, a site of $\mathbb{T}$ is open with probability p or close with probability $1 - p$, independently of the others.

This can also be seen as a random coloring (in black or white) of the faces of hexagonal lattice $\mathbb{T}^*$ dual of $\mathbb{T}$.

We define the exploration curve as follows. Let Ω be a simply connected subgraph of the triangular lattice and A,B be two points on its boundary. Then we can divide the hexagonal cells of $\partial\Omega$ into two arcs, going from A to B in two directions (clockwise and counter-clockwise). These arcs will be denoted by $\mathbb{B}$ and $\mathbb{W}$ such that $A, \mathbb{B}, B, \mathbb{W}$ is in the clockwise direction. Assume that all of hexagons in B are colored in black and all of hexagons in $\mathbb{W}$ are colored in white. The color of the hexagonal faces in Ω is chosen at random (black with probability p and white with probability $1-p$), independently of the others. We define the *exploration curve* γ starting at A and ending at B which separates the black component containing $\mathbb{B}$ from the white component containing $\mathbb{W}$.

Then the exploration curve γ is a self-avoiding walk using the vertices and edges of hexagonal lattice $\mathbb{T}^*$. We can define this interface γ by an equivalent way: It is a self-avoiding walk. At each step, γ look at its three neighbors on hexagonal lattice. One is occupied by the last step of γ. For the next step, γ chooses randomly one of these neighbors which have not yet occupied by γ. If there is just one neighbor which has not yet occupied, we choose this neighbor and if there are two neighbors, we choose the right neighbor with probability p and the left neighbor with probability $1 - p$.

We know that there exists $p_c \in [0, 1]$ such that for $p < p_c$ there is almost surely no infinite cluster, while for $p > p_c$ there is almost surely an infinite cluster. This parameter is called *critical point*. We can see in [2], the critical point of site-percolation on the triangular lattice equals $\frac{1}{2}$. The lower bound of critical point was proved by Harris in [8]. A similar theorem in the case of bond percolation on square lattice by Kesten in [10].

For this section and the next section, we take $\Omega = \mathbb{T}_+^*$. The hexagons on the boundary of Ω ($\partial\Omega$) and on the right of origin (denoted by $\partial^+\Omega$) are colored in black and the hexagons on $\partial\Omega$ and on the left of origin ($\partial^-\Omega$) are colored in white. In this case, the exploration curve is an (random) infinite self-avoiding walk. Denoted by $T_{\mathbb{T}_+^*}$ be the self-avoiding tree is constructed from the self-avoiding walks in $\mathbb{T}_+^*$. For simplify, we set $T = \mathbb{T}_+^*$.

In the case $\lambda = +\infty$, by the construction of the exploration curve, the limit walk ω^∞ on TSL_T has the same law with the exploration curve γ. Then all the properties of γ is until valid for ω^∞ of RW_∞. One of these properties is γ reaches the boundary of Ω an infinite times almost surely. This property is until valid in the case RW_λ, for all $\lambda > \lambda_c$ (see theorem 6.3 below).

6.3. **An inequality.**

Theorem 6.1. *Let $\lambda > \lambda_c$ and A be a subset of $\Omega \setminus \{0\}$ which is surrounded by a simple curve γ, and such that $\Omega \setminus A$ is simply connected. We denote ω^∞ (respectively $\omega^{\infty,A}$) the limit walk* *of RW_λ for the tree T_Ω (respectively $T_{\Omega\setminus A}$). We denote by $\mathbb{P}$ (respectively $\widetilde{\mathbb{P}}$) the associated probability measure, then*

$$\mathbb{P}(\omega^\infty \cap \gamma \neq \varnothing) \geqslant \widetilde{\mathbb{P}}(\omega^{\infty,A} \cap \gamma \neq \varnothing).$$

Proof. We prove this theorem by a coupling method. Recall a family of random variables $(U_x)_x$ in the section 2.3. We set $W_0 = 0, \widetilde{W_0} = 0$ almost surely. Assume that we constructed $(W_0, W_1, \dots, W_n)$ and $(\widetilde{W_0}, \widetilde{W_1}, \dots, \widetilde{W_n})$. We construct W_{n+1} and $\widetilde{W}_{n+1}$ as follow:

- If $W_n \notin \partial A$, we set:
$$W_{n+1} = U_{W_n} \text{ and } \widetilde{W}_{n+1} = W_{n+1}.$$
- If $W_n \in \partial A$ and $\widetilde{W}_n \in \partial A$, let Y be a random variable of uniform law on $[0,1]$ and $C(\lambda, T_\Omega^{W_n})$ (respectively $C(\lambda, T_{\Omega\setminus A}^{W_n})$) is the effective conductance of parameter λ on the tree $T_\Omega^{W_n}$ (respectively $T_{\Omega\setminus A}^{W_n}$). By the corollary 2.10, we obtain
$$C(\lambda, T_\Omega^{W_n}) \geqslant C(\lambda, T_{\Omega\setminus A}^{W_n}).$$
We take randomly a value of Y (uniform law):
  - If $Y \in \Big[0, C(\lambda, T_{\Omega\setminus A}^{W_n})\Big]$, for all $k \geqslant n+1$, we set
$$W_k = \Upsilon, \widetilde{W_k} = \Upsilon.$$
  - If $Y \in \Big[C(\lambda, T_{\Omega\setminus A}^{W_n}), C(\lambda, T_\Omega^{W_n})\Big]$, we set
$$W_{n+1} = W_{n-1} \text{ and for all } k \geqslant n+1 : \widetilde{W_k} = \Upsilon.$$
  - If $Y \in \Big[C(\lambda, T_\Omega^{W_n}), 1\Big]$, we set:
$$W_{n+1} = W_{n-1} \text{ and } \widetilde{W}_{n+1} = \widetilde{W}_{n-1}.$$
- If $W_n \in \partial A$ and $\widetilde{W_n} = \Upsilon$, let Y be a random variable of uniform law on $[0,1]$ and $C(\lambda, T_{\Omega\setminus A}^{W_n})$ is the effective conductance of parameter λ on the tree $T_{\Omega\setminus A}^{W_n}$. We take randomly a value of Y (uniform law):
  - If $Y \in \Big[0, C(\lambda, T_{\Omega\setminus A}^{W_n})\Big]$, for all $k \geqslant n+1$, we set:
$$W_k = \Upsilon$$
  - If $Y \in \Big[C(\lambda, T_{\Omega\setminus A}^{W_n}), 1\Big]$,
$$W_{n+1} = W_{n-1}.$$

Let μ (respectively ν) be the law of random walk $(W_n)_n$ (respectively $(\widetilde{W}_n)_n$). We can see that

$$\widetilde{\mathbb{P}}(\omega^{\infty,A} \cap \gamma \neq \varnothing) = \mu(\exists n > 0, W_n = \Upsilon) \text{ and } \mathbb{P}(\omega^\infty \cap \gamma \neq \varnothing) = \nu(\exists n > 0, \widetilde{W_n} = \Upsilon).$$

By the construction of W_n and $\widetilde{W_n}$, we have

$$\mu(\exists n > 0, W_n = \Upsilon) \leqslant \nu(\exists n > 0, \widetilde{W_n} = \Upsilon).$$

This completes the proof of theorem. □

Remark 6.2. *The random walks W_n and $\widetilde{W_n}$ are not the biased symmetric random walks. Then the coupling in the proof of theorem 6.1 is not a coupling between the biased symmetric random walks on T_Ω and $T_{\Omega\setminus A}$. The theorem 6.1 is until valid if we replace a subset A and a curve γ by any subset B and C such that if we want to reach B, we must be through C.*

6.4. **Some property of $\mathbb{P}_\lambda$ and $\mathbb{Q}_\lambda$.** We will prove the following theorem:

Theorem 6.3. *For all $\lambda > \lambda_c$, under the $\mathbb{P}_\lambda$ (or $\mathbb{Q}_\lambda$) measure, the infinite self-avoiding walk reaches the line Ox (i.e the line $\mathbb{Z} \otimes \{0\}$) an infinity of times almost surely.*

In order to prove the theorem 6.3, we need the following function:

$$\Pi : x \in V(T) \mapsto x_{|x|} \in \mathbb{Z}^2 \text{ where } T = T_{\mathbb{H}} \text{ or } T = T_{\mathbb{Z}^2}.$$

The proof of theorem 6.3 consists of several steps. The first step, we study the trajectory of the biased random walk X_n. We prove that, under the $\mathbb{P}_\lambda$-measure and $\mathbb{Q}_\lambda$-measure, it reaches the line Ox almost surely. In the second step, we prove that it reaches the line Ox an infinite number of times almost surely. The third step, we prove that under $\mathbb{Q}_\lambda$ -measure, the limit walk reaches the line Ox an infinite number of times almost surely and the last step we prove that under $\mathbb{P}_\lambda$ -measure, the limit walk reaches the line Ox an infinite number of times almost surely.

The first step. In this step, we study the trajectory of RW_λ. We begin with the following lemma:

Lemma 6.4. *Let $\lambda > \lambda_c$, we consider the biased symmetric random walk RW_λ on $T_{\mathbb{Z}^2}$ or $T_{\mathbb{H}}$. Then* $\limsup |\Re(\Pi(X_n))| = +\infty$ *almost surely.*

Proof. We prove the lemma in the case $T_{\mathbb{H}}$. In the same way, we obtain the result for $T_{\mathbb{Z}^2}$.

Assume that $p = \mathbb{P}(\limsup |\Re(\Pi(X_n))| < +\infty) > 0$, thus $1 - p = \mathbb{P}(\limsup |\Re(\Pi(X_n))| = +\infty) < 1$, then

$$\exists n_0 > 0 : \mathbb{P}(-n_0 \leqslant \Re(\Pi(X_n)) \leqslant n_0, \forall n) = q > 0.$$

For each $i \geqslant 0$, we set $T^i := \inf\{n : \Im(\Pi(X_n)) = i\}$. It is easy to see that $T^i < +\infty$ on the event $\{-n_0 \leqslant \Re(\Pi(X_n)) \leqslant n_0, \forall n\}$, because the biased random walk is transient. We remark that, at time T^i, we can always go towards the left or the right. We define:

$$S_i := \{\exists! k : |\Re(\Pi(X_k))| = n_0 + 1, \Im(X_k) = i \text{ and } \forall n \neq k : -n_0 \leqslant \Re(\Pi(X_n)) \leqslant n_0\}.$$

If the walk is at time T^i, we go towards the left or the right to reach the domain

$$\{\Re z = n_0 + 1\} \bigcup \{\Re z = -n_0 - 1\},$$

and after, we go back to X_{T^i}. We need at most $2n_0$ steps to do it. Then, there exist a constant $c > 0$ such that $\mathbb{P}(S_i) \geqslant cq, \forall i$.

Moreover, $\bigcup_{i=0}^{+\infty} S_i \subset \{|\Re X(X_n)| \leqslant n_0 + 1, \forall n\}$ and these S_i are disjoints. Then,

$$\mathbb{P}(\{|\Re(\Pi(X_n))| \leqslant n_0 + 1, \forall n\}) \geqslant \sum \mathbb{P}(S_i) = +\infty$$

This is a contradiction and then $\limsup |\Re(X_n)| = +\infty$ almost surely. ☐

Theorem 6.5. *Let $\lambda > \lambda_c$. We consider the biased random walk RW_λ on $T_{\mathbb{Z}^2}$ or $T_{\mathbb{H}}$. Then,*

$$|\{n > 0 : \Im(\Pi(X_n)) = 0\}| \geqslant 1,$$

almost surely.

Proof. To simplify, we set $Y_n := \Pi(X_n)$. We will prove this theorem by contradiction. First, we prove that $|\{n > 0, \Im Y_n = 0\}| \geqslant 1$ a.s for the tree $T_{\mathbb{Z}^2}$.

We set: $p = \mathbb{P}(\forall n > 0, \Im(Y_n) > 0)$. Assume that $p > 0$, then

$$\mathbb{P}(\exists n > 0, \Im(Y_n) = 0) = 1 - p < 1$$

We write $\{\exists n > 0 : \Im(Y_n) = 0\} = \bigcup_{n=1}^{+\infty} \{\exists 0 < k \leqslant n : \Im(Y_k = 0)\}$.

As the sequence $(\{k \leqslant n : \Im(Y_k) = 0\})_n$ is an increasing sequence. Thus,

$$\mathbb{P}(\exists n : \Im(Y_n) = 0) = \lim_n \mathbb{P}(\exists k \in (0, n] : \Im(Y_k) = 0).$$

Let $\varepsilon > 0$, there exist then n_0 such that: $\forall n \geqslant n_0, \mathbb{P}(\exists k \in (0, n] : \Im(Y_k) = 0) \geqslant 1 - p - \varepsilon$.

We know that the biased random walk does not reach the line Ox with a probability $p > 0$. By lemma 6.4, the random walk X_n must reach the domain $H := \{\Re(z) = n_0\} \bigcup \{\Re(z) = -n_0\}$

with a probability 1. We consider the first time S, that the random walk X_n reaches H and we assume that it reaches the line $\{\Re(z) = n_0\}$. We go on the random walk one step to reach the line $\{\Re(z) = n_0 + 1\}$. We consider a new half-plane at origin Y_S. We have also the random walk after the time S will stay in this half-plan with a probability p. Thus,

$$\mathbb{P}(\forall k \leqslant n_0 : \Im(Y_k) > 0 \text{ and } \exists k > n_0 : \Im(Y_k) = 0) = \frac{\lambda p^2}{1 + 3\lambda}.$$

As two events

$$\{\forall k \leqslant n_0 : \Im(Y_k) > 0; \exists k > n_0 : \Im(Y_k) = 0)\} \text{ and } \{\exists k \in (0, n_0] : \Im(Y_k) = 0\}$$

are disjoint and they are included in the event $\{\exists n > 0 : \Im(Y_n) = 0\}$, then:

$$\mathbb{P}(\{\exists n > 0 : \Im(Y_n) = 0\}) = 1 - p \geqslant 1 - p - \varepsilon + \frac{\lambda p^2}{1 + 3\lambda}.$$

If we take ε small enough, we will find a contradiction. It implies $p = 0$.

Now, we prove that $|\{n : \Im(Y_n) = 0\}| \geqslant 1$ a.s for the tree $T_{\mathbb{H}}$. We set $p = \mathbb{P}(\exists n > 0 : \Im(Y_n) = 0)$. Assume that $p > 0$, as the random walk in the domain $\{\Im(z) > 0\}$ of half-plane is the same law with the random walk in this domain of the plan. It implies that the random walk X_n on the plan does not reach the line Ox with a positive probability. That is a contradiction with the step 1 and then $p = 0$. □

The second step. The goal of this step is to prove the following theorem:

Theorem 6.6. *Let $\lambda > \lambda_c$. We consider the biased symmetric random walk RW_λ on $T_{\mathbb{Z}^2}$ or $T_{\mathbb{H}}$. Then,*

$$|\{n > 0 : \Im(Y_n) = 0\}| = +\infty,$$

almost surely.

Proof of theorem 6.6 in the case $T_{\mathbb{H}}$. Denote by A the following event:

$$A := \{|\{n > 0 : \Im Y_n = 0\}| = \infty\}.$$

We can write:

$$A = \{\forall k, \exists n > k : \Im Y_n = 0\}.$$

Suppose that $\mathbb{P}(A) < 1$, we have then $\mathbb{P}(A^c) > 0$. We write the event A^c under the form:

$$A^c := \{\exists k, \forall n > k : \Im Y_n > 0\}.$$

We set $A^c_m := \{\forall n > m : \Im Y_n > 0\}$, then $A^c = \bigcup A^c_n$.

We can see that it is an increasing union, thus $\exists n_0 : \mathbb{P}(A^c_{n_0}) > 0$ and then $\mathbb{P}(\forall n > n_0 : \Im X_n > 0) = c > 0$.

Now, we consider the random walk until time n_0. Denote by Ω_{n_0} the set of all configurations $(Y_0, Y_1, \dots, Y_{n_0})$. For each $\omega \in \Omega_{n_0}$, we define the event A_ω as follows: The random walk does not reach the line Ox after time n_0 and $(Y_0, Y_1, \dots, Y_{n_0}) = \omega$, then

$$\sum_{\omega \in \Omega_{n_0}} \mathbb{P}(A_\omega) = c > 0.$$

As the cardinal of Ω_{n_0} is finite, then there exists $\omega_0 \in \Omega_{n_0}$ such that: $\mathbb{P}(A_\omega) = c_1 > 0$.

We add a new line under the line Ox and we consider a new half-plane $\mathbb{H}'$ with the origin at O' (see the Figure 4).

We consider the biased random walk RW_λ on $T_{\mathbb{H}'}$. We consider the random walk begin at O', it comes to O in the first step, and after it does a trajectory as the configuration ω_0, we obtain a configuration ω'_0 (see the figure 2). The random walk on $T_{\mathbb{H}}$ and $T_{\mathbb{H}'}$ do not have the same law until the time n_0, but as n_0 fix, therefore the configuration $A_{\omega'_0}$ occurs with a positive probability, where $A_{\omega'_0}$ is defined as the same way with A_{ω_0}. This means that the biased random walk on $T_{\mathbb{H}'}$ does not reach the line $O'x$ with a positive probability. It is a contradiction and then $\mathbb{P}(A) = 1$, that concludes the proof of theorem 6.6 in the case $T_{\mathbb{H}}$.

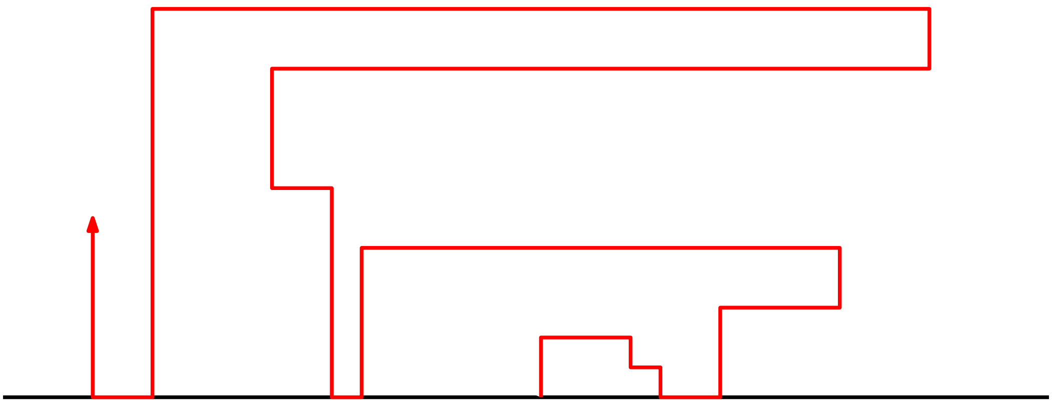

Figure 3. A configuration ω

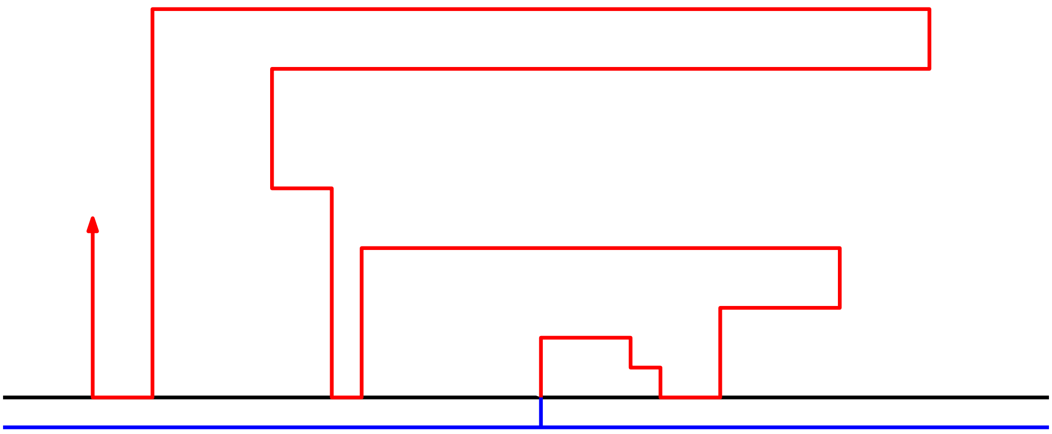

Figure 4. A new half-plane $\mathbb{H}'$

Proof of theorem 6.6 in the case $T_{\mathbb{Z}^2}$. Suppose that the random walk reaches the line Ox an infinite number of times with a probability strictly greater than 1.

By using the same argument as in the case $T_{\mathbb{H}}$, then there exists a configuration ω and a number n_0 such that $\mathbb{P}(A_\omega) > 0$ where A_ω is an event that the random walk does not reach the line Ox after the time n_0 and $(Y_0, Y_1, \ldots, Y_{n_0}) = \omega$ (see Figure 5).

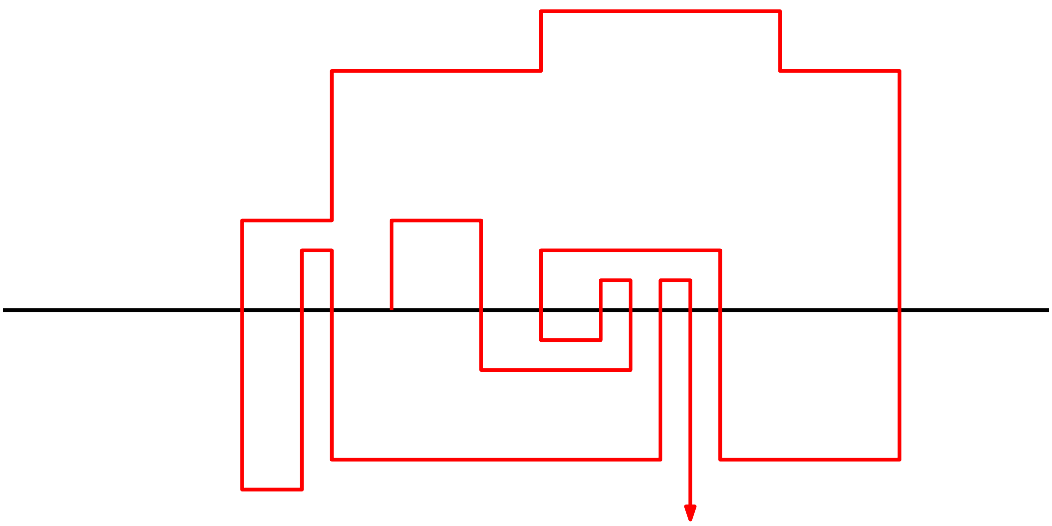

Figure 5. A configuration of A_ω

Let $A_1 = (a_1, 0), B_1 = (b_1, 0) \dots, A_k = (a_k, 0), B_k = (b_k, 0)$ be k point of intersections of the line $\mathbb{Z} \times \{0\}$ with ω along the curve ω such that there exists a self-avoiding walk C_{a_i} in ω starting at $(a_i, 0)$ and ending at $(b_i, 0)$ which is below the line $\mathbb{Z} \times \{0\}$. Let $(a, 0)$ be the last point of intersection of the line $\mathbb{Z} \times \{0\}$ with ω before that the random walk does not reach the line $\mathbb{Z} \times \{0\}$. We set $A := (x, 0)$ be one of $(a_i, 0)$ or $(b_i, 0)$ which maximize the first coordinate. We take $B = (x, 1)$ and we consider a new plan $\mathbb{P}$ with the origin at B as in the figure 4, and we consider the random walk RW_λ on the tree $T_{\mathbb{HP}}$ from the self-avoiding walks in the half-plane of the plan $\mathbb{P}$. Let $C = (C_{a_1}, C_{a_2}, \dots, C_{a_l})$ be a set of l self-avoiding walks in ω which connect $(a_i, 0)$ to $(b_i, 0)$. If there exist i, j such that $[a_j \wedge b_j, a_j \vee b_j] \subset [a_i \wedge b_i, a_i \vee b_i]$, then we remove the self-avoiding walk C_{a_j} in C. Finally, we obtain a set of self-avoiding walk $C' = (C'_{a_1}, C'_{a_2}, \dots, C'_{a_m})$ which there are not i, j such that $[a_j \wedge b_j, a_j \vee b_j] \subset [a_i \wedge b_i, a_i \vee b_i]$ and we can assume that $C' = (C_{a_1}, C_{a_2}, \dots, C_{a_m})$ and that $a_1 > a_2 > \cdots > a_m$ and for all $i \in [1, m]$, $a_i < b_i$, this hypothesis implies that $A \equiv (b_1, 0)$. We consider the configuration ω' as follows (see the Figure 6):

Set $u = \sup\{1 \leqslant i \leqslant m : a_i > a\}$. Then we define the self-avoiding walk

$$\begin{cases} \omega'_1 := ([BA], C_{a_1}, [(b_2, 0), (a_1, 0)], C_{a_2}, [(b_3, 0), (a_2, 0)], \dots, C_{a_u}, [(a_u, 0)(a_u, 1)]) \\ \omega'_2 := ([(a_u, 1), (a_m, 1)], [(a_m, 1), (a_m, 0)], C_{a_m}, [(b_m, 0), (a_{m-1}, 0)] \dots, C_{a_{u+1}}, [(b_{u+1}, 0), (a, 0)]) \\ \omega'_3 := \omega_{|[t, n_0]} \text{ where } \omega(t) = (a, 0) \end{cases}$$

We take $\omega' := \omega'_1 \oplus \omega'_2 \oplus \omega'_3$.

Then, we can see that the random walk reach a finite number of times the half-plane of $\mathbb{P}$ with a strictly positive probability. This is a contradiction with the case $T_{\mathbb{H}}$ that we proved in the first case. $\square$

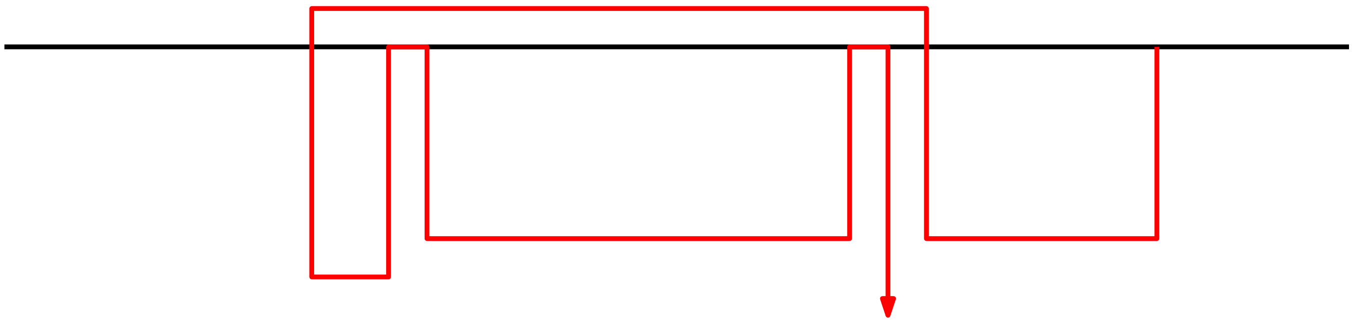

FIGURE 6. The configuration ω'

Remark 6.7. *All of results that we proved in the first step and second step for $T_{\mathbb{Z}^2}$ and $T_{\mathbb{H}}$, are still valid for $TSL_{\mathbb{H}}$ and $TSL_{\mathbb{Z}^2}$. We can see that it is suffice to prove the theorem 6.3 in the case $TSL_{\mathbb{H}}$ and $TSL_{\mathbb{Z}^2}$, (it means the biased random walk on $TSL_{\mathbb{H}}$ and $TSL_{\mathbb{Z}^2}$ reach the line Ox an infinite number of times).*

The third step. In this step, we give a proof of theorem 6.3 in the case $\mathbb{Q}_\lambda$-measure. We start with the following definition

Definition 6.8. Let C be a closed, simple curve of $\mathbb{Z}^2$. The interior of C, denoted by $I(C)$ is a sub-domain of $\mathbb{R}^2$ which is surrounded by C (see the figure 7). Denote by $S(C)$ is the area of this domain. The exterior of C is defined by

$$E(C) := \mathbb{R}^2 \setminus I(C).$$

Lemma 6.9. *Let $((a_1, 0), (a_2, 0), \dots, (a_{2n,0}))$ be a sequence of points on the line $\mathbb{Z} \times \{0\}$ such that $a_1 < a_2 < \cdots < a_{2n}$. For each i, we denote γ_i the self-avoiding walk starting at $(a_{2i-1}, 0)$ and ending at $(a_{2i}, 0)$ which is below the line $\mathbb{Z} \times 0$. Suppose that*

$$\forall i, \gamma_i \cap \gamma_j = \varnothing.$$

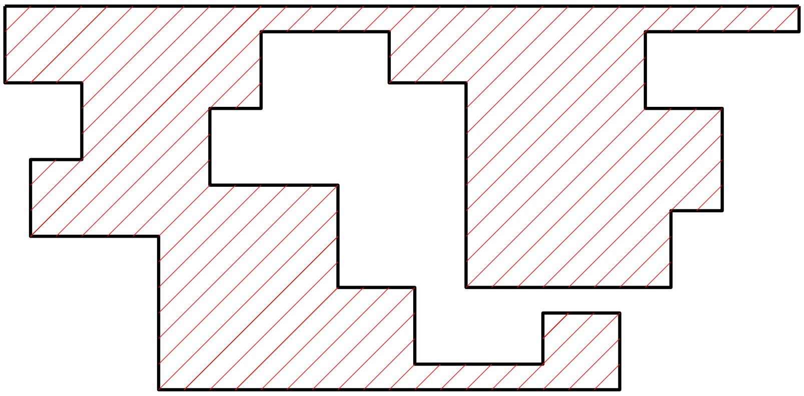

FIGURE 7. A closed, simple curve C of $\mathbb{Z}^2$ with its interior in red

We set $A := \bigcup \gamma_i$ *and* $B = \partial A \cup ((\cup_{i=1}^{n} [a_{2i-1}, a_{2i}]) \times \{0\})$ *where,*

$$\partial A := \left\{ z \in \mathbb{Z}^2 : \exists x \in A, 0 < d(x,z) \leqslant \sqrt{2} \right\}$$

Then there exists a self-avoiding walk in B *starting at* $(a_1 - 1, 0)$ *and ending at* $(a_{2n+1}, 0)$.

Proof. The statement is intuitively clear, but its proof is a little messy; we postpone it to Appendix A. □

Remark 6.10. *By the same argument as in the proof of Lemma 6.9 (see Appendix A), we can see that: Given* $v \in \mathbb{Z} \times \{0\}$ *such that there exists an infinite self-avoiding walk which is below the line* $\mathbb{Z} \times \{0\}$ *starting at* v*, there exists a self-avoiding walk in* B *starting at* v *and ending at* $(a_{2n} + 1, 0)$.

Proof of theorem 6.3 in the case $\mathbb{Q}_\lambda$*-measure.* Denote by A the following event:

$$A := \{ |\{ n > 0 : \Im \omega_\lambda^\infty(n) = 0 \}| = \infty \} .$$

We can write A under the form:

$$A := \{ \forall k, \exists n > k : \Im \omega_\lambda^\infty(n) = 0 \} .$$

Assume that $\mathbb{P}(A) < 1$, then $\mathbb{P}(A^c) > 0$. By the symmetry, the following event has a positive probability

$$A_1^c := \{ \exists k, \forall n > k : \Im \omega_\lambda^\infty(n) < 0 \}$$

We set $A_m^c := \{ \forall n > m : \Im \omega_\lambda^\infty(n) < 0 \}$, then $A_1^c = \bigcup A_n^c$.

We can see that it is an increasing union, thus $\exists n_0 : \mathbb{P}(A_{n_0}^c) > 0$ and then $\mathbb{P}(\forall n > n_0 : \Im \omega_\lambda^\infty(n) < 0) = c > 0$.

Now we consider the limit walk until the step n_0, $(\omega_\lambda^\infty(0), \omega_\lambda^\infty(1), \dots, \omega_\lambda^\infty(n_0))$. By using the same argument as in the second step, there exist $\omega(0), \omega(1), \dots, \omega(n_0)$ such that the following event has a strictly positive probability (see Figure 8):

$$B := \begin{cases} \omega_\lambda^\infty(0) = \omega(0), \omega_\lambda^\infty(1) = \omega(1), \dots, \omega_\lambda^\infty(n_0) = \omega(n_0) \\ \forall n > n_0 : \Im \omega_\lambda^\infty(n) < 0 \end{cases}$$

We fix $k > 0$ and define:

$$D := (\{\Im z \geqslant 0\} \cap \left\{ \Re z < \inf_{0 \leqslant i \leqslant n_0} \Re \omega_\lambda^\infty(i) \right\}) \cup (\{\Im z \geqslant 0\} \cap \left\{ \Re z > \sup_{0 \leqslant i \leqslant n_0} \Re \omega_\lambda^\infty(i) \right\})$$

Let V be a subset of $\mathbb{Z} \setminus D$ such that for all $x \in V$, there exists an infinite self-avoiding walk starting at x and it does not reach the path $(\omega_\lambda^\infty(0), \omega_\lambda^\infty(1), \dots, \omega_\lambda^\infty(n_0))$.

For each $x \in V$, we denote by γ_x the path starting at x, which does not reach the path $(\omega(0), \dots, \omega(n_0))$, and reaches the domain D at only one point and such that, for each $z \in \gamma_x$, z belong to the line $\mathbb{Z} \times \{0\}$ or z belong to the boundary of self-avoiding walk $(\omega(0), \omega(1), \dots, \omega(n_0))$. By the lemma 6.9 and the remark 6.10, γ_x as above exists. We set then $p := \sup_{x \in V} |\gamma_x|$.

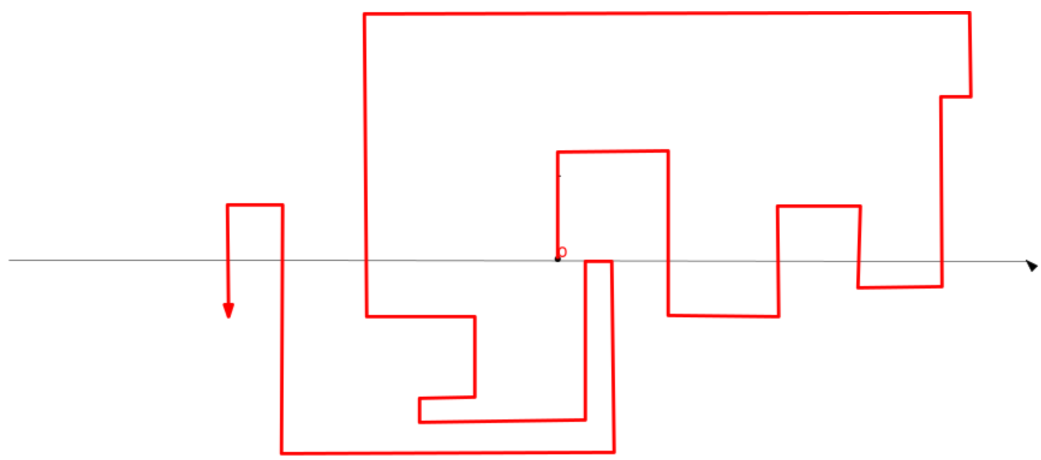

FIGURE 8. A configuration of B

Let $\overline{T}$ be a tree defined by:

$$\begin{cases} \left|\overline{T}_i\right| = 1 \text{ for all } i \leqslant p \\ \overline{T}_p = \{v\} \\ \overline{T}^v = T_{\mathbb{Q}} \end{cases}$$

We apply the theorem 6.6, almost surely, the random walk reaches the line Ox an infinite number of times, and thus, it reaches the line Ox at least k times almost surely. Every time it reaches the line Ox at a point x, we can go on the random walk at most p step to reach the domain D (we can do it, because $TSL_{\mathbb{Z}^2}$ have no leaf and then x belong to V). Then, the limit walk stays within the half-plane $\Im z < 0$ after the step n_0 with a probability smaller than $(1 - C(\lambda, \overline{T}))$, where $C(\lambda, \overline{T})$ is the effective conductance for the tree $\overline{T}$. In general, we have:

$$\forall k > 0 : \mathbb{P}(B) \leqslant (1 - C(\lambda, \overline{T}))^k$$

As we have $C(\lambda, \overline{T}) > 0$ (because it contains the tree $T_{\mathbb{Q}}$), then $\mathbb{P}(B) = 0$. That is a contradiction, and it implies the theorem 6.3 in the case $\mathbb{Q}_\lambda$-measure. □

The last step. In this section, we give a proof of theorem 6.3 in the case $\mathbb{P}_\lambda$-measure. We begin with the following:

Proposition 6.11. *We consider the biased random walk RW_λ on $TSL_{\mathbb{H}}$. Let $(B_n)_n$ be the sequence of strips of $\mathbb{H}$ where B_n is the strip between two lines $\Im z = 0$ and $\Im z = n$. We define $p_s := \lim_n br(T_{B_n})$, where T_{B_n} is a tree which is constructed from the self-avoiding walks in B_n. Suppose that $\lambda > \frac{1}{p_s}$. Then the limit walk ω_λ^∞ touch the line Ox an infinite number of times almost surely.*

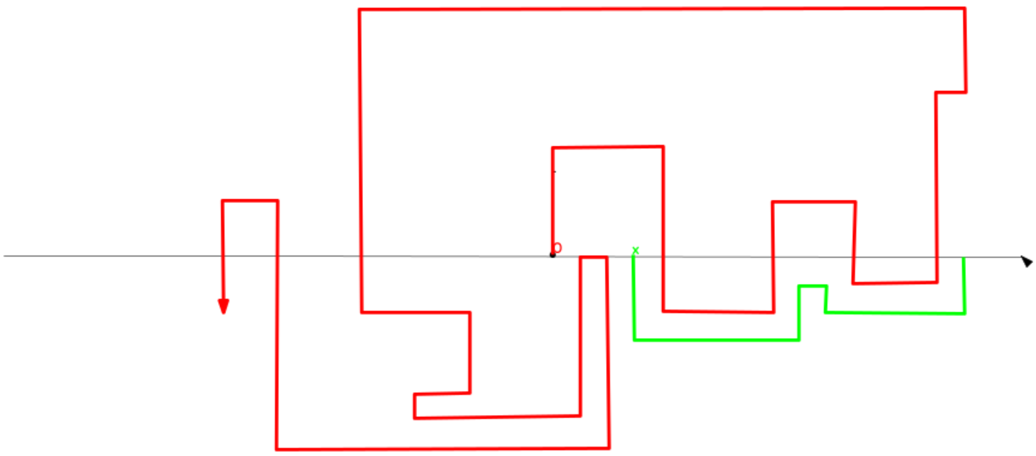

FIGURE 9. A configuration of B with a path γ_x

Proof. We fix $\lambda > p_s$. First, we prove that there exists $L > 0$ such that the limit walk on the tree $TSL_{\mathbb{H}}$ reaches the strip B_L almost surely.

Assume that, for all $L > 0$, the limit walk reaches the strip B_L a finite number of times with a strictly positive probability. We take L_0 such that $\lambda > \frac{1}{br(T_{B_{L_0}})}$. We use the same argument as in the second step, then there exists a sequence of vertices of $\mathbb{H}$, $\omega(0), \omega(1), \ldots, \omega(n_0)$ such that the following event has a strictly positive probability (see the figure):

$$B' := \begin{cases} \omega_\lambda^\infty(0) = \omega(0), \omega_\lambda^\infty(1) = \omega(1), \ldots, \omega_\lambda^\infty(n_0) = \omega(n_0) \\ \forall n > n_0 : \Im \omega_\lambda^\infty(n) > L_0 \end{cases}$$

We set $b := \sup_{0 \leqslant i \leqslant n_0} \Re \omega_\lambda^\infty(i) - \inf_{0 \leqslant i \leqslant n_0} \Re \omega_\lambda^\infty(i)$. Let $\widehat{T}$ be a tree defined by:

$$\begin{cases} \left|\widehat{T}_i\right| = 1 \text{ for all } t \leqslant b \\ \widehat{T}_b = v \\ \widehat{T}^v = T_{B_{L_0}} \end{cases}$$

By theorem 6.6, we know that the random walk reaches the line Ox an infinite number of times and then it must to reach the line $\Im z = L_0$ an infinite number of times almost surely. By using the same argument as in the third step, then:

$$\forall k > 0 : \mathbb{P}(B') \leqslant (1 - C(\lambda, T_{B_{L_0}}))^k$$

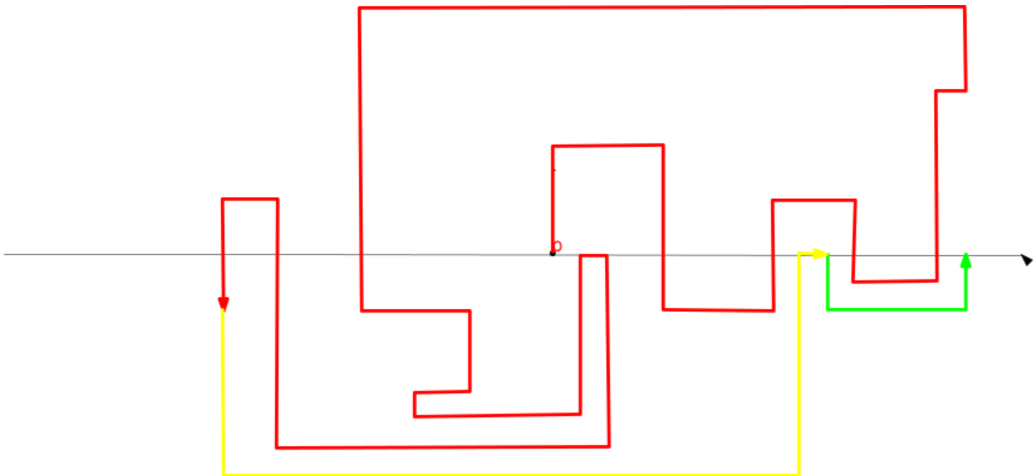

FIGURE 10. The random walk reaches the line $\mathbb{Z} \times \{0\}$ at z and it go to D by the path γ_z

As we have $C(\lambda, T_{B_{L_0}}) > 0$ (because we have taken $\lambda > \lambda_c(T_{B_{L_0}})$, then $\mathbb{P}(B') = 0$. That is a contradiction. We conclude that there exists $L > 0$ such that the limit walk on the tree $TSL_{\mathbb{H}}$ reaches the strip B_L almost surely.

We fix a number L such that the limit walk reaches the domain B_L an infinite number of times almost surely. Now, we prove that the limit walk reaches an infinite number of times the line Ox almost surely.

Assume that $\mathbb{P}(|n : \Im\omega^\infty(n) = 0 < +\infty|) > 0$, then there exists $n_0; z_1, z_2, \ldots, z_{n_0}$ such that the following event occurs with a strictly positive probability:

$$C := \begin{cases} \omega_\lambda^\infty(0) = 0; \omega_\lambda^\infty(1) = z_1; \ldots; \omega_\lambda^\infty(n_0) = z_{n_0} \\ \forall n > n_0 : \Im\omega_\lambda^\infty(n) > 0 \end{cases}$$

Let T^* be a tree defined by

$$\begin{cases} |T_i^*| = 1 \text{ for all } t \leqslant L \\ T_L^* = \{v\} \\ (T^*)^v = T_{B_L} \end{cases}$$

Recall that $Y_n := \Pi X_n$. Let U be a set of naturals n such that: $\Re Y_n = \sup_{0 \leqslant i \leqslant n; Y_i \in B_L} \Re Y_i$ or $\Re Y_n = \inf_{0 \leqslant i \leqslant n; Y_i \in B_L} \Re Y_i$. For each $n \in U$, we go on the walk in the vertical direction until it reaches the line Ox. When it reaches the line Ox, it remains reach the line Ox with a probability which is greater than $c \times C(\lambda, T^*)$ where c is a constant that does not depend on n (see the figure 11).

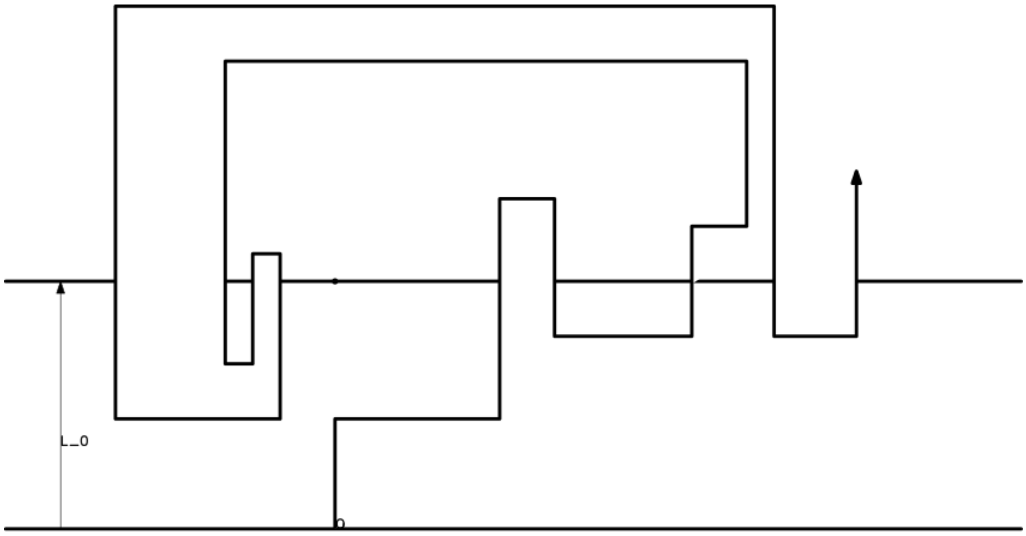

FIGURE 11. A configuration of B'

As the walk touches the line Ox an infinite number of times almost surely, we have then $|U| = +\infty, p.s.$ This implies that $\mathbb{P}(C) = 0$. That is a contradiction.

In order to complete the proof of Theorem 6.3, we must have $p_s = br(T_{\mathbb{H}})$. This equality is the content of Proposition 5.15. $\square$

Remark 6.12. *In the prove of the proposition 6.11, if $\mathbb{P}(B') = 0$, we have then $\mathbb{P}(B") > 0$, where*

$$B" := \begin{cases} \omega_\lambda^\infty(0) = \omega(0), \omega_\lambda^\infty(1) = \omega(1), \ldots, \omega_\lambda^\infty(n_0) = \omega(n_0) \\ \forall n > n_0 : \Im\omega_\lambda^\infty(n) < L_0 \end{cases}$$

By the same argument with the case $\mathbb{P}(B') > 0$, we can find a contradiction.

Theorem 6.13. *For all $\lambda > \lambda_c$, we have*

$$\mathbb{R}_\lambda(\limsup_n \Re\omega_\lambda^\infty(n) = +\infty) = 1; \mathbb{R}_\lambda(\liminf_n \Re\omega_\lambda^\infty(n) = -\infty) = 1,$$

where $\mathbb{R} \in \{\mathbb{P}, \mathbb{Q}\}$.

Proof of theorem 6.13 in the case $\mathbb{Z}^2$. Assume that $\mathbb{P}(\limsup_n \Re\omega_\lambda^\infty(n) = +\infty) < 1$, we have then $\mathbb{P}(\exists A > 0, \exists n_0 > 0, \forall n \geqslant n_0 : \Re\omega_\lambda^\infty(n) < A) > 0$.

Now we consider the limit walk until the step n_0, $(\omega_\lambda^\infty(0), \omega_\lambda^\infty(1), \ldots, \omega_\lambda^\infty(n_0))$. By using the same argument as in the proof of theorem 6.6, there exists $\omega(0), \omega(1), \ldots, \omega(n_0)$ such that the following event have a strictly positive probability:

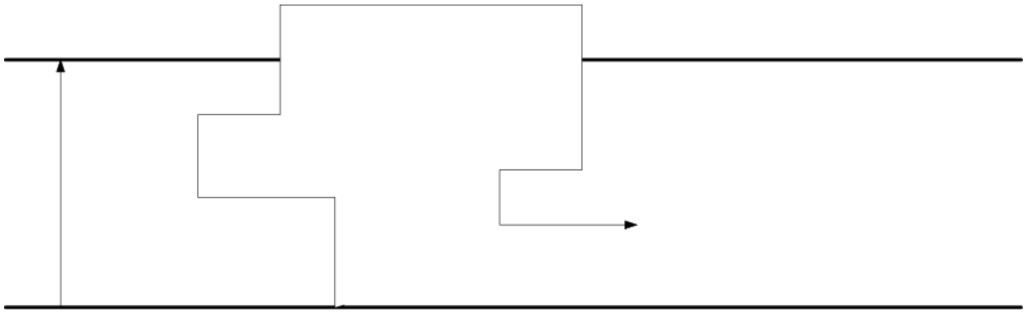

FIGURE 12.

$$M := \begin{cases} \omega_\lambda^\infty(0) = \omega(0), \omega_\lambda^\infty(1) = \omega(1), \dots, \omega_\lambda^\infty(n_0) = \omega(n_0) \\ \forall n > n_0 : \Re\omega_\lambda^\infty(n) < 0 \end{cases}$$

By the symmetry, the theorem 6.3 implies that the infinite self-avoiding walk touches the line Oy an infinite number of times almost surely. Then, we use the same argument as in the third step 6.4 to find a contradiction, and then $\mathbb{P}(\limsup_n \Re\omega_\lambda^\infty(n) = +\infty) = 1$. By the symmetry, we also have $\mathbb{P}(\liminf_n \Re\omega_\lambda^\infty(n) = -\infty) = 1$. □

Proof of theorem 6.13 in the case half-plane $\mathbb{H}$. We perform the procedure of the prove of theorem 6.3 to obtain: The trajectory of the biased random walk X_n reaches the line $\{0\} \times \mathbb{Z}^+$ almost surely, then it reaches the line $\{0\} \times \mathbb{Z}^+$ an infinite number of times almost surely. Finally, we use the same argument as in the third step 6.4 to conclude. □

Remark 6.14. *The theorems 6.3 and 6.13 are still valid in the other regular lattices such as hexagonal lattice, triangular lattice.*

6.5. **The law of first k-steps of limit walk.** We consider the biased random walk RW_λ on $T_{\mathbb{H}}$. Recall that ω_λ^∞ is the associated limit walk and $\mathbb{P}_\lambda$ denote its law.

Let $k \in \mathbb{N}^*$ and $y_1, y_2, \dots, y_k$ be k elements of $V(T_{\mathbb{H}})$ such that $(y_1, y_2, \dots, y_k)$ is a simple path. For each $\lambda > \lambda_c$, recall that the law of first k-steps is defined by:

$$\varphi^{\lambda,k}(y_1, y_2, \dots, y_k) = \mathbb{P}_\lambda(\omega_\lambda^\infty(1) = y_1, \omega_\lambda^\infty(2) = y_2, \dots, \omega_\lambda^\infty(k) = y_k). \tag{7}$$

We prove the continuity of this function.

Theorem 6.15. *For every $k \in \mathbb{N}^*$ and $(y_1, y_2, \dots, y_k) \in V^k$, the function $\varphi^{\lambda,k}$ is a continuous function of λ on $(\lambda_c, +\infty)$.*

We begin with some notation. Let T be a tree and $x \in V(T)$, recall that $C(\lambda, T^x)$ denote the effective conductance of biased random walk X_n^x on the tree T^x. If $y \in (T^x)_1$, we define

$$C(\lambda, T^x, y) := \mathbb{P}(\forall n > 0, X_n^x > x; X_1^x = y).$$

In order to prove the theorem 6.15, we need the following lemma:

Lemma 6.16. *We have*

$$\varphi^{\lambda,k}(y_1, y_2, \dots, y_k) = \frac{C(\lambda, T^0, y_1)}{\widetilde{C}(\lambda, T^0)} \cdot \frac{C(\lambda, T^{y_1}, y_2)}{\widetilde{C}(\lambda, T^{y_1})} \cdots \frac{C(\lambda, T^{y_{k-1}}, y_k)}{\widetilde{C}(\lambda, T^{y_{k-1}})}.$$

Proof of lemma 6.16. We prove this lemma in the cases $k = 1$ and $k = 2$. In the case $k \geqslant 3$, we use the same method.

The case $k = 1$ We let $C^i(\lambda, T)$ denote the probability return to origin k times before go to the infinite for the biased random walk on the tree T. We define the events $A := \{\omega_\lambda^\infty(1) = y_1\}$ and A_i denote the random walk return to origin k times before goes to the infinite by passing through y_1. In other words,

$$A_i := \{\omega_\lambda^\infty(1) = y_1; |n > 0 : X_n = o| = k\}.$$

The events A_i are disjoints, then we can see that

$$A = \bigcup_{i=0}^{+\infty} A_i.$$

We have $\mathbb{P}(A_0) = C(\lambda, T^o, y_1); \mathbb{P}(A_i) = C(\lambda, T^o, y_1) C^i(\lambda, T^o), \forall i \geqslant 1$, we obtain then

$$\mathbb{P}(A) = \sum_{i=0}^{+\infty} \mathbb{P}(A_i) = (1 + \sum_{i=1}^{+\infty} C^i(\lambda, T^o)) C(\lambda, T^o, y_1).$$

Moreover, by the Markov property, we have also

$$(1 + \sum_{i=1}^{+\infty} C^i(\lambda, T^o)) = \sum_{i=0}^{+\infty} (1 - \widetilde{C}(\lambda, T^o))^i = \frac{1}{\widetilde{C}(\lambda, T^o)}.$$

Then we obtain $\varphi^{\lambda,1}(y_1, y_2, \dots, y_k) = \mathbb{P}(A) = \frac{C(\lambda, T^o, y_1)}{\widetilde{C}(\lambda, T^o)}$.

The case $k = 2$ We define the events $A := \{\omega_\lambda^\infty(1) = y_1, \omega_\lambda^\infty(2) = y_2\}$ and A_{ij} denote the random walk returns to origin i times and after the last time that it returns to origin, it returns to y_1 j times. In other word,

$$A_{ij} := \{\omega_\lambda^\infty(1) = y_1; \omega_\lambda^\infty(2) = y_2; |n > 0 : X_n = o| = i; |n > S_i : X_n = y_1| = j\},$$

where S_i satisfy $X_{S_i} = o; |0 < n < S_i : X_n = o| = i - 1$. Then

$$A = \bigcup_{i=0}^{+\infty} \bigcup_{j=0}^{+\infty} A_{ij}.$$

The events A_{ij} are disjoints, thus

$$\mathbb{P}(A) = \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} \mathbb{P}(A_{ij}).$$

Moreover, it is easy to see that

$$\mathbb{P}(A_{ij}) = C^i(\lambda, T^0) \times C(\lambda, T^0, y_1) \times C^j(\lambda, T^{y_1}) \times C(\lambda, T^{y_1}, y_2).$$

Then we get

$$\mathbb{P}(A) = \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} C^i(\lambda, T^0) \times C(\lambda, T^0, y_1) \times C^j(\lambda, T^{y_1}) \times C(\lambda, T^{y_1}, y_2).$$

Then

$$\mathbb{P}(A) = \sum_{i=0}^{+\infty} C^i(\lambda, T^0) \times C(\lambda, T^0, y_1) \sum_{j=0}^{+\infty} C^j(\lambda, T^{y_1}) \times C(\lambda, T^{y_1}, y_2).$$

By the Markov property, we have

$$\sum_{j=0}^{+\infty} C^j(\lambda, T^{y_1}) = \frac{1}{\widetilde{C}(\lambda, T^{y_1})}; \sum_{j=0}^{+\infty} C^i(\lambda, T^0) = \frac{1}{\widetilde{C}(\lambda, T^0)}.$$

We get $\mathbb{P}(A) = \frac{C(\lambda,T^0,y_1)}{\widetilde{C}(\lambda,T^0)} \times \frac{C(\lambda,T^{y_1},y_2)}{\widetilde{C}(\lambda,T^{y_1})}$. □

Proof of theorem 6.15. By lemma 6.16, we have

$$\varphi^{\lambda,k}(y_1, y_2, \ldots, y_k) = \frac{C(\lambda, T^0, y_1)}{\widetilde{C}(\lambda, T^0)} \times \frac{C(\lambda, T^{y_1}, y_2)}{\widetilde{C}(\lambda, T^{y_1})} \times \cdots \times \frac{C(\lambda, T^{y_{k-1}}, y_k)}{\widetilde{C}(\lambda, T^{y_{k-1}})}.$$

It is enough to prove that $C(\lambda, T^{y_i}, y_{i+1})$ and $C(\lambda, T^{y_i})$ are continuous. For the continuity of $C(\lambda, T^{y_i})$, we use the same method as in the proof of theorem 5.16. For the continuity of $C(\lambda, T^{y_i}, y_{i+1})$, this function can be written in terms of λ and $C(\lambda, T^{y_{i+1}})$. It implies the continuity of this function. □

Remark 6.17. *Theorem 6.15 is still valid in the case $T_{\mathbb{Z}^2}$. Beside, let $k \in \mathbb{N}^*$ and $z \in V(T_{\mathbb{H}})$. We define*

$$\varphi(\lambda, k, z) := \mathbb{P}_\lambda(\omega_\lambda^\infty(k) = z).$$

Then $\varphi(\lambda, k, z)$ is a continuous function on $(\frac{1}{\mu}, +\infty)$. Indeed, we can write

$$\varphi(\lambda, k, z) = \sum_{\gamma \in SAW; \gamma(0)=0, \gamma(k)=z} \frac{C(\lambda, T^0, \gamma(1))}{\widetilde{C}(\lambda, T^0)} \times \frac{C(\lambda, T^{\gamma(1)}, \gamma(2))}{\widetilde{C}(\lambda, T^{\gamma(1)})} \times \cdots \times \frac{C(\lambda, T^{\gamma(k-1)}, z)}{\widetilde{C}(\lambda, T^{\gamma(k-1)})}.$$

7. The critical probability measure through biased random walk

7.1. **The critical probability measure.** In this section, $\mathbb{H}$ is the upper-half plane and we now consider the self-avoiding tree T defined by either $T = T_{\mathbb{H}}$ or $T = T_{\mathbb{Z}^2}$. We aim to construct a critical probability measure through the biased random walk on self-avoiding tree. First, we review the construction of Madras and Slade (see [14] for detail). Recall that b_n be the number of all n-step bridges that begin at 0 and B_n denote the set of all n-step bridges that begin at 0. Given $n \geqslant m$ and an m-step self-avoiding walk γ in $\mathbb{H}$. Let $\mathbb{P}^B_{m,n}(\gamma)$ denote the fraction of n-step bridges that extend γ, it means

$$\mathbb{P}^B_{m,n}(\gamma) = \frac{|F_n(\gamma) \cap B_n|}{b_n} = \frac{|F_n(\gamma)|}{b_n}, \tag{8}$$

where $F_n(\gamma)$ is the set of all n-step bridges which extend γ. The equality (8) is the probability that a long bridge (uniformly chosen from among all n-step bridges) is an extension of γ. Define

$$\mathbb{P}^B_m(\omega) := \lim_{n\to\infty} \mathbb{P}^B_{m,n}(\gamma). \tag{9}$$

Fact 7.1 ([14], Theorem 8.3.1)**.** *Let γ be an m-step self-avoiding walk in $\mathbb{H}$. Then the limit (9) exists.*

The existence of the measures $\mathbb{P}^B_m$ allows us to define a measure $\mathbb{P}^B_\infty$ on the set SAW_∞ of $\mathbb{H}$. For each $\gamma^\infty \in SAW_\infty$, denote $\gamma^\infty\,[0, m]$ be $\gamma^\infty(0), \gamma^\infty(1), \ldots, \gamma^\infty(m)$, then

$$\mathbb{P}^B_\infty(\gamma^\infty\,[0, m] = \gamma) = \mathbb{P}^B_m(\gamma), \text{ for every } \gamma.$$

Fact 7.2 ([14], Theorem 8.3.2)**.** *$\mathbb{P}^B_\infty$ is the $\frac{1}{\mu}$-Kesten measure, where μ is the connective constant of the square lattice.*

We state now the main theorem of this section.

Theorem 7.3. *Recall that for all $m \geqslant 1$, T^m is the m-good tree. Fix $k \geqslant 1$ and $y_1, y_2, \dots, y_k \in V(T_{\mathbb{H}})$, the function $\varphi^{m,\lambda,k}(y_1,\dots,y_k)$ (respectively $\varphi^{\mathbb{H},\lambda,k}(y_1,\dots,y_k)$) denote the law of first k-steps of RW_λ on T^m (respectively $T_{\mathbb{H}}$). Then,*

1. *The function $\varphi^{m,\lambda,k}(y_1,\dots,y_k)$ converges towards a limit, denoted by $\varphi^{m,\lambda_m,k}(y_1,\dots,y_k)$ when λ decreases towards $\lambda_m = \lambda_c(T^m)$.*
2. *The function $\varphi^{m,\lambda_m,k}(y_1,\dots,y_k)$ converges towards a limit, denoted by $\varphi^{\lambda_c,k}(y_1,\dots,y_k)$.*
3. *Moreover, we have the following diagram*

$$\begin{array}{ccc}
\varphi^{m,\lambda,k}(y_1,\dots,y_k) & \xrightarrow[\lambda>\lambda_c(T_{\mathbb{H}})]{m\to+\infty} & \varphi^{\mathbb{H},\lambda,k}(y_1,\dots,y_k) \\
{\scriptstyle \lambda\to\lambda_c(T^m)}\Big\downarrow & & \Big\downarrow{\scriptstyle ?} \\
\varphi^{m,\lambda_m,k}(y_1,\dots,y_k) & \xrightarrow[m\to+\infty]{} & \varphi^{\lambda_c,k}(y_1,\dots,y_k)
\end{array}$$

Proof of points 1 and 2 of Theorem 7.3. It is suffices to prove the theorem in the case $k = 2$ and we use the same method for all $k \geqslant 3$. By fact 3.8, we have

$$\text{for all } i \in \{1,2,3\} \lim_{\lambda\to\lambda_c(T^m)} \varphi^{m,\lambda,2}(y_0 = 0, y_1 = x_i) = \sum_{\gamma\in S^i} \lambda_m^{|\gamma|},$$

where S^i is a set of all irreducible bridges which pass through x_i and $\lambda_c(T^m) = \lambda_m$. Let $p_{i,n}$ be the number of irreducible bridges of length n which are pass through x_i, we obtain

$$\lim_{\lambda\to\lambda_c(T^m)} \varphi^{m,\lambda,2}(y_0 = 0, y_i) = \sum_{n=1}^{m} p_{i,n}\lambda_m^n.$$

This implies that $\varphi^{m,\lambda_m,2}(y_0 = 0, y_1 = x_i) = \sum_{n=1}^m p_{i,n}\lambda_m^n$. Moreover, for all $m, \lambda_m \geqslant \lambda_c(= \lambda_c(T_{\mathbb{H}}))$ since $T^m \subset T_{\mathbb{H}}$. Then,

$$\varphi^{m,\lambda_m,2}(y_0 = 0, y_1 = x_i) \geqslant \sum_{n=1}^{m} p_{i,n}\lambda_c^n.$$

We need to prove that $\varphi^{m,\lambda_m,2}(y_0 = 0, y_1 = x_i)$ converges. Assume that there exists a subsequence $(m_k)_k$ such that

$$\begin{cases} \forall i \in \{1,2,3\}, \lim_{k\to+\infty} \varphi^{m_k,\lambda_{m_k},2}(y_0 = 0, y_1 = x_i) = \alpha_i \\ \alpha_1 > \sum_{n=1}^{+\infty} p_{1,n}\lambda_c^n \end{cases}$$

We can see that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and for all $i \in \{1,2,3\}, \alpha_i \geqslant \sum_{n=1}^{+\infty} p_{1,n}\lambda_c^n$. By Fact 5.5, we have $\sum_{i=1}^3 \sum_{n=1}^{+\infty} p_{i,n}\lambda_c^n = 1$ and then $1 = \alpha_1 + \alpha_2 + \alpha_3 > \sum_{i=1}^3 \sum_{n=1}^{+\infty} p_{i,n}\lambda_c^n = 1$. That is a contradiction. It implies that for all $i \in \{1,2,3\}, \alpha_i = \sum_{n=1}^{+\infty} p_{i,n}\lambda_c^n$. We conclude that $\varphi^{m,\lambda_m,2}(y_0 = 0, y_1 = x_i)$ converges towards $\sum_{n=1}^{+\infty} p_{i,n}\lambda_c^n$ when $m \to +\infty$. □

Proof of point 3 of Theorem 7.3. It remains to prove that

$$\lim_{m\to+\infty,\lambda>\lambda_c(T_{\mathbb{H}})} \varphi^{m,\lambda,k}(y_1,\dots,y_k) = \varphi^{\mathbb{H},\lambda,k}(y_1,\dots,y_k).$$

It is suffices to prove the theorem in the case $k = 2$, the same method will extend for all $k \geqslant 3$.

Fix $\lambda > \lambda_c(T_{\mathbb{H}})$ and $\varepsilon > 0$. By the proof of proposition 5.10, we have $\lim_{m\to+\infty} \lambda_c(T^m) = \lambda_c(T_{\mathbb{H}})$. Then there exists $m_0 > 0$ such that,

$$\begin{cases} \forall m \geqslant m_0, \lambda > \lambda_c(T^m) \\ \forall m \geqslant m_0, (1 - C(\lambda, T^m))^m < \varepsilon \end{cases}$$

Let T be a tree defined by:

$$\begin{cases} |T_i| = 1 \text{ for all } i \leqslant m \\ T_p = \{v\} \\ T^v = T^m \end{cases}$$

We choose n_0 (depends on m) such that for all $n > n_0$, $(1 - C(\lambda))^n < \varepsilon$. By considering the self-avoiding walks in the rectangle whose vertices are $((-n_0, 0); (-n_0, m_0); (n_0, m_0); (n_0, 0))$ and a simple argument, we can see that for all $i > m_0 n_0$, $\left|\varphi^{i,\lambda,k}(y_1,\dots,y_k) - \varphi^{\mathbb{H},\lambda,k}(y_1,\dots,y_k)\right| < 2\varepsilon$. Since ε is arbitrary, this complete the proof of theorem. $\square$

Remark 7.4.
- *Theorem 7.3 allows us to define a critical probability measure $\mathbb{P}_{\lambda_c}$ on $T_{\mathbb{H}}$. We can see that this critical probability measure is exactly Kesten's measure as in Section 5.2.*
- *Let $T(m) = T_{\mathbb{Z}^2,m}$ be the finite tree which is construct from self-avoiding walks starting at origin of length smaller than m. Fix $k \geqslant 1$ and $y_1, y_2, \dots, y_k \in V(T_{\mathbb{Z}^2})$, recall that $T(m)^{\infty,T(m)}$ is $T(m)$-finite type and $\varphi^{m,\lambda,k}(y_1,\dots,y_k)$ is the law of first k steps of RW_λ on $T(m)^{\infty,T(m)}$. Then the function $\varphi^{m,\lambda,k}(y_1,\dots,y_k)$ converges towards a limit, denoted by $\varphi^{m,\lambda_m,k}(y_1,\dots,y_k)$ when λ decreases towards $\lambda_m = \lambda_c(T(m)^{\infty,T(m)})$. We hope that the function $\varphi^{m,\lambda_m,k}(y_1,\dots,y_k)$ converges when $m \to \infty$.*

7.2. **Conjectures.** If we take the cut-set $\Pi_n := T_n$ and we set $c(e) = (\frac{1}{\mu})^{|e|}$, then

$$\sum_n \Big(\sum_{e\in\Pi_n} c(e)\Big)^{-1} = \sum_{n=1}^{+\infty} \frac{\mu^n}{c_n}.$$

If the prediction of Nienhuis holds, we obtain

$$\sum_{n=1}^{+\infty} \frac{\mu^n}{c_n} \geqslant c \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{11}{32}}} = +\infty$$

By Theorem 2.14, we can establish the following conjecture.

Conjecture 7.5. *The biased random walk RW_{λ_c} on $T_{\mathbb{H}}$ (or $T_{\mathbb{Z}^2}$) is recurrent.*

Finally, we believe that for every $k \geqslant 1$ and $y_1, y_2, \dots, y_k \in V(T_{\mathbb{H}})$,

$$\lim_{\lambda\to\lambda_c(T_{\mathbb{H}})} \varphi^{\mathbb{H},\lambda,k}(y_1,\dots,y_k) = \varphi^{\lambda_c,k}(y_1,\dots,y_k).$$

Conjecture 7.6. *The following convergence diagram holds*

$$\begin{tikzcd}
\varphi^{m,\lambda,k}(y_1,\dots,y_k) \arrow[r, "m\to+\infty", "\lambda>\lambda_c(T_{\mathbb{H}})"'] \arrow[d, "\lambda\to\lambda_c(T^m)"'] & \varphi^{\mathbb{H},\lambda,k}(y_1,\dots,y_k) \arrow[d, "\lambda\to\lambda_{\mathbf{c}}(\mathbf{T}_{\mathbb{H}})"] \\
\varphi^{m,\lambda_m,k}(y_1,\dots,y_k) \arrow[r, "m\to+\infty"'] & \varphi^{\lambda_c,k}(y_1,\dots,y_k)
\end{tikzcd}$$

Appendix A. Proof of Lemma 6.9

We give here an algorithm to prove Lemma 6.9. By a recursive argument, it is suffices to prove lemma in the case $n = 1$. Let $\omega := \gamma_1$ and ω' be a self-avoiding walk in $\mathbb{Z}^2 \setminus \omega$, starting at $(a_1 - 1, 0)$ and ending at $(a_2 + 1, 0)$. Then we obtain a polygon $\omega\omega'$ (see the figure 5).

We follow the following algorithm:

INPUT: Two self-avoiding walk ω and ω'.

OUTPUT: A self-avoiding walk γ which is in B.

1. Let $\gamma := \omega', U := \{x \in \gamma \setminus B\}$

2. We choose an element $z = (a, b)$ of U

Case 1: If the edge $(z, (a, b+1)) \in \gamma$ and the square $(z, (a, b+1), (a+1, b+1), (a+1, b)) \in \omega\gamma$. We remove the edge $(z, (a, b+1))$ and we add 3 edges $((a, b+1), (a+1, b+1)); ((a+1, b+1), (a+1, b)); ((a, b+1), (a+1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$.

Case 2: If the edge $(z, (a, b+1)) \in \gamma$ and the square $(z, (a, b+1), (a-1, b+1), (a-1, b)) \in \omega\gamma$. We remove the edge $(z, (a, b+1))$ and we add 3 edges $((a, b+1), (a-1, b+1)); ((a-1, b+1), (a-1, b)); ((a, b+1), (a-1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$.

Case 3: If the edge $(z, (a, b-1)) \in \gamma$ and the square $(z, (a, b-1), (a+1, b-1), (a+1, b)) \in \omega\gamma$. We remove the edge $(z, (a, b-1))$ and we add 3 edges $((a, b-1), (a+1, b-1)); ((a+1, b-$

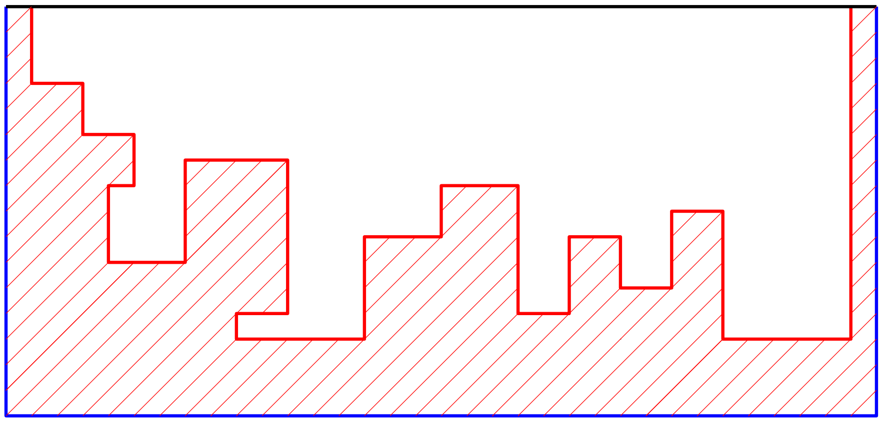

FIGURE 13. The self-avoiding walk ω (in red) and ω' (in blue) and the polygon between ω and ω'.

$1), (a+1, b)); ((a, b-1), (a+1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$.

Case 4: If the edge $(z, (a, b-1)) \in \gamma$ and the square $(z, (a, b-1), (a-1, b-1), (a-1, b)) \in \omega\gamma$. We remove the edge $(z, (a, b-1))$ and we add 3 edges $((a, b-1), (a-1, b-1)); ((a-1, b-1), (a-1, b)); ((a, b-1), (a-1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$.

Case 5: If the edge $(z, (a+1, b)) \in \gamma$ and the square $(z, (a+1, b), (a+1, b-1), (a, b-1)) \in \omega\gamma$. We remove the edge $(z, (a+1, b))$ and we add 3 edges $((a+1, b), (a+1, b-1)); ((a+1, b-1), (a, b)); ((a, b+1), (a+1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$.

Case 6: If the edge $(z, (a+1, b)) \in \gamma$ and the square $(z, (a+1, b), (a+1, b+1), (a, b+1)) \in \omega\gamma$. We remove the edge $(z, (a+1, b))$ and we add 3 edges $((a+1, b), (a+1, b+1)); ((a+1, b+1), (a, b+1)); ((a, b+1), (a+1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$.

Case 7: If the edge $(z, (a-1, b)) \in \gamma$ and the square $(z, (a-1, b), (a-1, b-1), (a, b-1)) \in \omega\gamma$. We remove the edge $(z, (a+1, b))$ and we add 3 edges $((a-1, b), (a-1, b-1)); ((a-1, b-1), (a, b-1)); ((a, b-1), (a-1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$.

Case 8: If the edge $(z, (a-1, b)) \in \gamma$ and the square $(z, (a-1, b), (a-1, b+1), (a, b+1)) \in \omega\gamma$. We remove the edge $(z, (a+1, b))$ and we add 3 edges $((a-1, b), (a-1, b+1)); ((a-1, b+1), (a, b+1)); ((a, b+1), (a-1, b))$ to γ. We choose a new self-avoiding walk τ in γ starting at $(a_1 - 1, 0)$ and ending at $(a_2 + 1, 0)$.

3. Let $\gamma := \tau; U := \{x \in \gamma \setminus B\}$.
4. While $U \neq \emptyset$ go back to 2.
5. Return γ.

We can see that whenever U is not empty, we can decrease the area of the interior of the polygon $\omega\gamma$. Indeed, for example if we are in the case 5, we have seven possibilities as in the figure 14. The first possibility (the left picture in Figure 14), the area of the interior of the polygon $\omega\gamma$ decrease 1. The second possibility (the right picture in Figure 14), the area of interior of the polygon $\omega\gamma$ decrease at least 1 (the blue zone). Similarly, the possibilities in Figures 15, 16 and 17, the area of interior of the polygon $\omega\gamma$ decrease at least 1.

Since the ere of the initial polygon is finite, then we can find a self-avoiding walk which is in B, starting at $(a_1 - 1, 0)$ and ending at $(a_n + 1, 0)$ after a finite steps of algorithm.

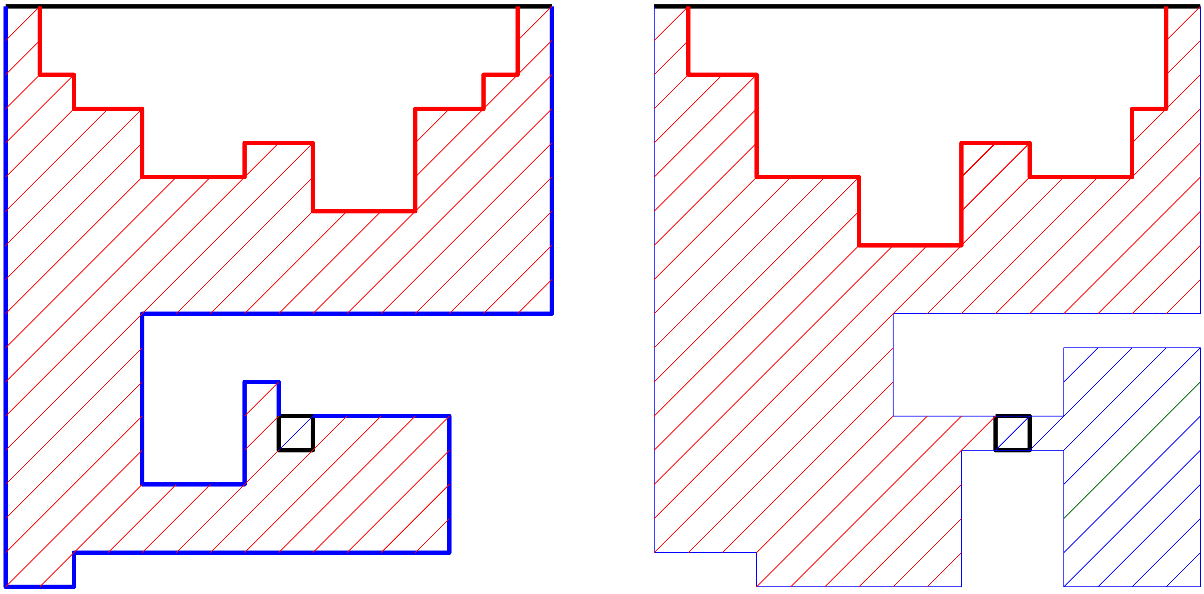

FIGURE 14. The self-avoiding walk ω (in red) and ω' (in blue and the polygon between ω and ω').

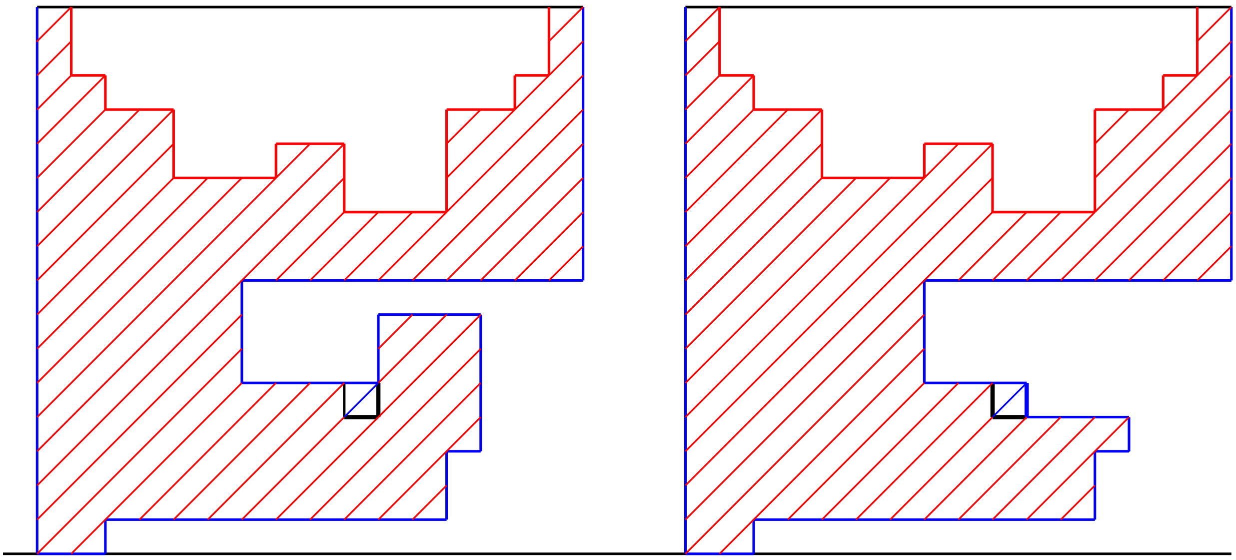

FIGURE 15. The self-avoiding walk ω (in red) and ω' (in blue and the polygon between ω and ω').

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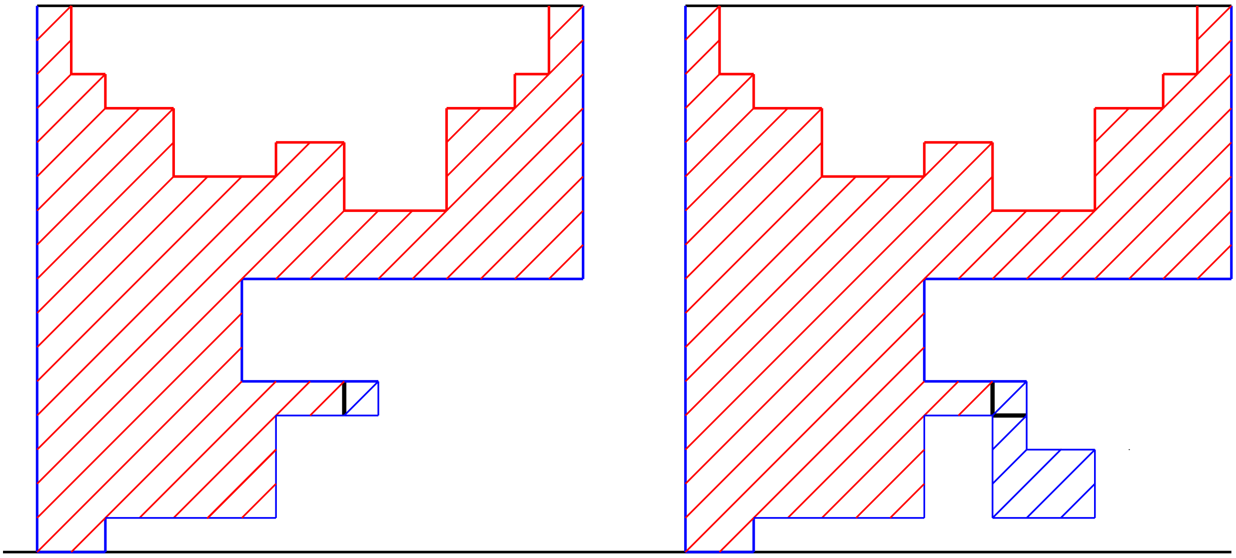

FIGURE 16. The self-avoiding walk ω (in red) and ω' (in blue and the polygon between ω and ω').

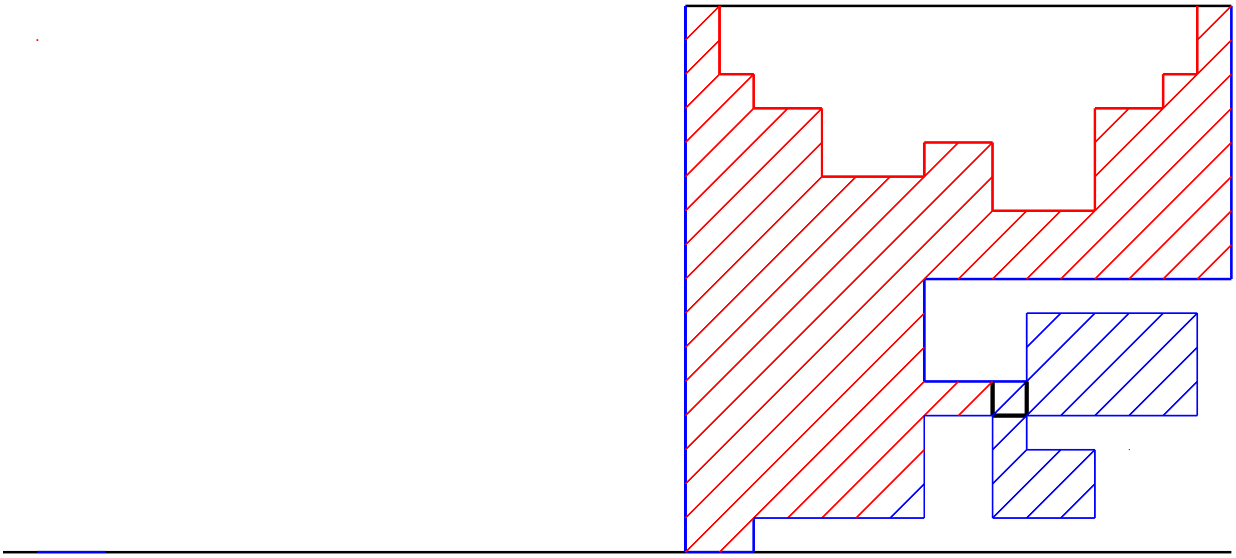

FIGURE 17. The self-avoiding walk ω (in red) and ω' (in blue and the polygon between ω and ω').

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