On the Uniqueness of Global Multiple SLEs

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Abstract

This article focuses on the characterization of global multiple Schramm-Loewner evolutions (SLE). The chordal SLE process describes the scaling limit of a single interface in various critical lattice models with Dobrushin boundary conditions, and similarly, global multiple SLEs describe scaling limits of collections of interfaces in critical lattice models with alternating boundary conditions. In this article, we give a minimal amount of characterizing properties for the global multiple SLEs: we prove that there exists a unique probability measure on collections of pairwise disjoint continuous simple curves with a certain conditional law property. As a consequence, we obtain the convergence of multiple interfaces in the critical Ising and FK-Ising models.

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1 Introduction

At the turn of the millennium, O. Schramm introduced random fractal curves in the plane which he called "stochastic Loewner evolutions" (SLE) [Sch00, RS05], and which have since then been known as Schramm-Loewner evolutions. He proved that these probability measures on curves are the unique ones that enjoy the following two properties: their law is conformally invariant, and, viewed as growth processes (via Loewner's theory), they have the domain Markov property — a memorylessness property of the growing curve. These properties are natural from the physics point of view, and in many cases, it has been verified that interfaces in critical planar lattice models of statistical physics converge in the scaling limit to SLE type curves, see [Smi01, LSW04, CN07, SS09, CS12, CDCH$^+$14] for examples.

In the chordal case, there is a one-parameter family (SLE_κ) of such curves, parameterized by a real number $\kappa > 0$ that is believed to be related to universality classes of the critical models, and the central charges of the corresponding conformal field theories. In this article, we consider several interacting SLE_κ curves, multiple SLEs. We prove that when $\kappa \in (0, 4]$, there exists a unique multiple SLE_κ measure on families of curves with a given connectivity pattern, as detailed in Theorem 1.2. Such measures have been considered in many works [BBK05, Dub07, Gra07, KL07, Law09], but we have not found a conceptual approach in the literature, in terms of a minimal number of characterizing properties in the spirit of Schramm's classification.

Results of convergence of a single discrete interface to an SLE curve in the scaling limit are all rather involved. On the other hand, after our characterization of the multiple SLEs, it is relatively straightforward to extend these convergence results to prove that multiple interfaces also converge to the multiple SLE_κ. Indeed, the relative compactness of the interfaces in a suitable topology can be verified with little effort using results in [CDCH16, DCST17, KS17], and the main problem is then to identify the limit (i.e., to prove that the subsequential limits are given by a unique collection of random curves).

As an application, we prove that the chordal interfaces in the critical Ising model with alternating boundary conditions converge to the multiple SLE_κ with parameter $\kappa = 3$, in the sense detailed in Sections 1.2 and 4.1. In contrast to the previous work [Izy17] of K. Izyurov, we work on the global collection of curves and condition on the event that the interfaces form a given connectivity pattern — see also Figure 1.1 for an illustration. We also identify the marginal law of one curve in the scaling limit as a weighted chordal SLE_3. As an input in our proof, together with results from [CDCH16, DCST17, KS17] for the relative compactness, we also use the convergence of a single critical Ising interface to the chordal SLE_3 from [CS12, CDCH$^+$14].

The explicit construction of the global multiple SLEs given in [KL07, Law09, PW17] and in Section 3 of the present article fails for $\kappa > 4$. Nevertheless, we discuss in Section 4 how, using information from discrete models, one could extend the classification of the multiple SLEs of our Theorem 1.2 to the range $\kappa \in (4, 6]$. More precisely, we prove that the convergence of a single interface in the critical random-cluster model combined with relative compactness implies the existence and uniqueness of a multiple SLE_κ, where $\kappa \in (4, 6]$ is related to the cluster weight q by Equation (4.7). In the special case of the FK-Ising model ($q = 2$), using the results of [CS12, CDCH$^+$14, CDCH16, DCST17, KS17], we obtain the convergence of any number of chordal interfaces to the unique multiple $\mathrm{SLE}_{16/3}$. However, for general $\kappa \in (4, 6)$, this result remains conditional on the convergence of a single interface (we note that the case $\kappa = 6$ corresponds to critical percolation, where the convergence is also known [Smi01, CN07]).

1.1 Global Multiple SLEs

Throughout, we let $\Omega \subset \mathbb{C}$ denote a simply connected domain with $2N$ distinct points $x_1, \ldots, x_{2N} \in \partial\Omega$ appearing in counterclockwise order along the boundary. We call the $(2N + 1)$-tuple $(\Omega; x_1, \ldots, x_{2N})$ a *(topological) polygon*. We consider curves in Ω each of which connects two points among $\{x_1, \ldots, x_{2N}\}$. These curves can have various planar (i.e., non-crossing) connectivities. We describe the connectivities by planar pair partitions $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$, where $\{a_1, b_1, \ldots, a_N, b_N\} = \{1, 2, \ldots, 2N\}$. We

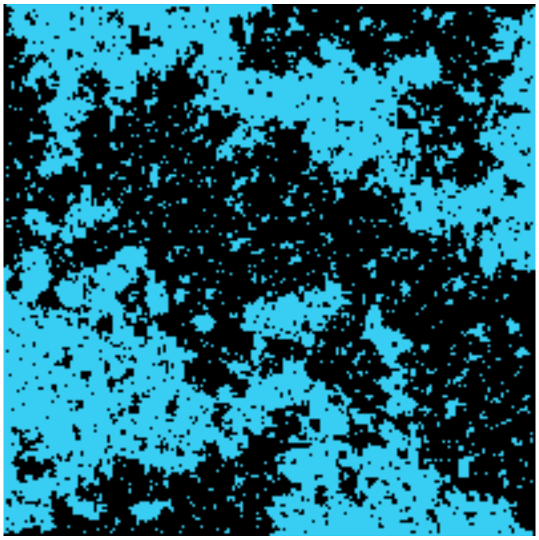

Figure 1.1: Simulation of the critical Ising model with alternating boundary conditions. There are eight marked points on the boundary of the polygon Ω and therefore, four interfaces connecting the marked points pairwise. We only illustrate one possible connectivity of the curves (the reader may verify that there are 14 different topological possibilities).

call such α *(planar) link patterns* and we denote the set of them by LP_N. Given a link pattern $\alpha \in \mathrm{LP}_N$ and $\{a, b\} \in \alpha$, we denote by $\alpha/\{a, b\}$ the link pattern in LP_{N-1} obtained by removing $\{a, b\}$ from α and then relabeling the remaining indices so that they are the first $2(N-1)$ integers.

We let $X_{\mathrm{simple}}(\Omega; x_1, x_2)$ denote the set of continuous simple unparameterized curves in Ω connecting x_1 and x_2 such that they only touch the boundary $\partial\Omega$ in $\{x_1, x_2\}$. When $\kappa \in (0, 4]$, the chordal SLE_κ curve belongs to this space almost surely. Also, when $N \geq 2$, we let $X^\alpha_{\mathrm{simple}}(\Omega; x_1, \ldots, x_{2N})$ denote the set of families $(\eta_1, \ldots, \eta_N)$ of pairwise disjoint curves, where $\eta_j \in X_{\mathrm{simple}}(\Omega; x_{a_j}, x_{b_j})$ for all $j \in \{1, \ldots, N\}$.

Definition 1.1. Let $\kappa \in (0, 4]$. For $N \geq 2$ and for any link pattern $\alpha \in \mathrm{LP}_N$, we call a probability measure on the families $(\eta_1, \ldots, \eta_N) \in X^\alpha_{\mathrm{simple}}(\Omega; x_1, \ldots, x_{2N})$ *a global N-SLE_κ associated to α* if, for each $j \in \{1, \ldots, N\}$, the conditional law of the curve η_j given $\{\eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_N\}$ is the chordal SLE_κ connecting x_{a_j} and x_{b_j} in the component of the domain $\Omega \setminus \bigcup_{i \neq j} \eta_i$ containing the endpoints x_{a_j} and x_{b_j} of η_j on its boundary.

Theorem 1.2. *Let $\kappa \in (0, 4]$ and let $(\Omega; x_1, \ldots, x_{2N})$ be a polygon with $N \geq 1$. For any $\alpha \in \mathrm{LP}_N$, there exists a unique global N-SLE_κ associated to α.*

The existence part of Theorem 1.2 is already known — see [KL07, Law09, PW17]. We briefly review the construction in Section 3.1. The uniqueness part of Theorem 1.2 for $N = 2$ was proved in [MS16b, Theorem 4.1], where the authors used a coupling of the SLE and the Gaussian free field. Unfortunately, this proof does not apply in general for N commuting SLEs, which is the case of the present article. We first give a different proof for the uniqueness when $N = 2$ by a Markov chain argument (in Section 3.2), and then generalize it for all $N \geq 3$ (in Section 3.3).

We note that it follows immediately from Definition 1.1 that global multiple SLEs have the following cascade property. Suppose $(\eta_1, \ldots, \eta_N) \in X^\alpha_{\mathrm{simple}}(\Omega; x_1, \ldots, x_{2N})$ has the law of a global multiple N-SLE_κ associated to the link pattern $\alpha \in \mathrm{LP}_N$. Assume also that $\{j, j+1\} \in \alpha$ for some $j \in \{1, \ldots, N\}$. Then, the conditional law of $(\eta_2, \ldots, \eta_N)$ given η_1 is a global $(N-1)$-SLE_κ associated to $\alpha/\{j, j+1\}$.

1.2 Multiple Interfaces in Critical Planar Ising Model

Next, we consider critical Ising interfaces in the scaling limit. Assuming that Ω is bounded, we let discrete domains $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ on the square lattice approximate $(\Omega; x_1, \ldots, x_{2N})$ as $\delta \to 0$ (we will provide the details of the approximation scheme in Section 4), and we consider the critical Ising model (which we also define in Section 4) on Ω^δ with the following *alternating boundary conditions* (see Figure 1.1):

$$\oplus \text{ on } (x_{2j-1}^\delta \, x_{2j}^\delta), \quad \text{for } j \in \{1, \ldots, N\}; \qquad \ominus \text{ on } (x_{2j}^\delta \, x_{2j+1}^\delta), \quad \text{for } j \in \{0, 1, \ldots, N\}, \tag{1.1}$$

with the convention that $x^\delta_{2N} = x^\delta_0$ and $x^\delta_{2N+1} = x^\delta_1$. Then, N interfaces $(\eta^\delta_1, \ldots, \eta^\delta_N)$ connect the $2N$ boundary points $x^\delta_1, \ldots, x^\delta_{2N}$, forming a planar connectivity encoded in a link pattern $\mathcal{A}^\delta \in \mathrm{LP}_N$.

To understand the scaling limit of the interfaces, we must specify the topology in which the convergence of the curves as $\delta \to 0$ occurs. Thus, we let X denote the set of planar oriented curves, that is, continuous mappings from $[0, 1]$ to $\mathbb{C}$ modulo reparameterization. More precisely, we equip X with the metric

$$d(\eta_1, \eta_2) := \inf \sup_{t\in[0,1]} |\eta_1(\varphi_1(t)) - \eta_2(\varphi_2(t))|, \tag{1.2}$$

where the infimum is taken over all increasing homeomorphisms $\varphi_1, \varphi_2 \colon [0, 1] \to [0, 1]$. Then, the metric space (X, d) is complete and separable.

Proposition 1.3. *Let $\alpha \in \mathrm{LP}_N$. Then, as $\delta \to 0$, conditionally on the event $\{\mathcal{A}^\delta = \alpha\}$, the law of the collection $(\eta^\delta_1, \ldots, \eta^\delta_N)$ of critical Ising interfaces converges weakly to the global N-SLE_3 associated to α. In particular, as $\delta \to 0$, the law of a single curve η^δ_j in this collection connecting two points x_j and x_k converges weakly to a conformal image of the Loewner chain with driving function given by Equation* (3.16) *in the end of Section 3, with $\kappa = 3$.*

We prove Proposition 1.3 in Section 4.1, where we also define the Ising model and discuss some of its main features. The key ingredients in the proof are [CS12, CDCH+14, KS17] and Theorem 1.2. In addition, we also need sufficient control on six-arm events and an appropriate Russo-Seymour-Welsh bound proved in [CDCH16] in order to rule out certain undesired behavior of the interfaces.

1.3 Multiple Interfaces in Critical Planar FK-Ising Model

Finally, we consider critical FK-Ising interfaces in the scaling limit. More generally, in Section 4 we study the random-cluster model, whose interfaces conjecturally converge to SLE_κ curves with $\kappa \in (4, 6]$. We define these models in Section 4.2. We consider the critical FK-Ising model in Ω^δ with the following *alternating boundary conditions* (see Figure 4.2):

$$\text{wired on } (x^\delta_{2j-1}\, x^\delta_{2j}), \quad \text{for } j \in \{1, \ldots, N\}; \qquad \text{free on } (x^\delta_{2j}\, x^\delta_{2j+1}), \quad \text{for } j \in \{0, 1, \ldots, N\}, \tag{1.3}$$

with the convention that $x^\delta_{2N} = x^\delta_0$ and $x^\delta_{2N+1} = x^\delta_1$. As in the case of the Ising model, N interfaces $(\eta^\delta_1, \ldots, \eta^\delta_N)$ connect the $2N$ boundary points $x^\delta_1, \ldots, x^\delta_{2N}$, forming a planar connectivity encoded in a link pattern $\mathcal{A}^\delta \in \mathrm{LP}_N$. However, this time the scaling limits are not simple curves, and we need to extend the definition of a global multiple SLE_κ to include the range $\kappa \in (4, 6]$. For this, we let $X_0(\Omega; x, y)$ denote the closure of the space $X_{\mathrm{simple}}(\Omega; x, y)$ in the metric topology of (X, d). Note that the curves in $X_0(\Omega; x, y)$ may have multiple points but no self-crossings. In particular, the chordal SLE_κ curve belongs to this space almost surely for all $\kappa > 4$.

Now, for $N \geq 2$ and $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$, we denote by $X^\alpha_0(\Omega; x_1, \ldots, x_{2N})$ the collection of curves $(\eta_1, \ldots, \eta_N)$ such that, for each $j \in \{1, \ldots, N\}$, we have $\eta_j \in X_0(\Omega; x_{a_j}, x_{b_j})$ and η_j has a positive distance from the arcs $(x_{a_j+1}\, x_{b_j-1})$ and $(x_{b_j+1}\, x_{a_j-1})$. Note that $X^\alpha_0(\Omega; x_1, \ldots, x_{2N})$ is not complete. In Definition 1.1, the global N-SLE_κ was defined for $\kappa \leq 4$ — we can now extend this definition to all $\kappa \in (0, 8)$ by replacing $X^\alpha_{\mathrm{simple}}(\Omega; x_1, \ldots, x_{2N})$ with $X^\alpha_0(\Omega; x_1, \ldots, x_{2N})$ in Definition 1.1. We remark that this definition would still formally make sense in the range $\kappa \geq 8$, but since for such values of κ, the SLE_κ process is described by a Peano curve, uniqueness of multiple SLE clearly fails, as one can specify the common boundaries of the different curves in an arbitrary way while preserving the conditional distributions of individual curves.

Proposition 1.4. *Theorem 1.2 also holds for $\kappa = 16/3$, and for any $\alpha \in \mathrm{LP}_N$, as $\delta \to 0$, conditionally on the event $\{\mathcal{A}^\delta = \alpha\}$, the law of the collection $(\eta^\delta_1, \ldots, \eta^\delta_N)$ of critical FK-Ising interfaces converges weakly to the global N-$\mathrm{SLE}_{16/3}$ associated to α.*

We prove Proposition 1.4 in Section 4.3. To show that the scaling limit is a global multiple SLE, we again use results from the literature [CS12, CDCH+14, KS17] combined with a Russo-Seymour-Welsh bound proved in [DCST17] and six-arm estimates. To prove the uniqueness of the limit, we use a Markov chain argument similar to the proof of Theorem 1.2.

Remark 1.5. *Combining the same argument that we use in Section 4.3 with the results of [Smi01, CN07] one can check that there also exists a unique global multiple* SLE_κ *for* $\kappa = 6$ *with any given connectivity pattern, and Proposition 1.4 holds for the critical site percolation on the triangular lattice with* $\kappa = 6$.

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2 Preliminaries

In this section, we give some preliminary results, for use in the subsequent sections. In Section 2.1, we discuss Brownian excursions and the Brownian loop measure. These concepts are needed frequently in this article. In Sections 2.2 and 2.3, we define the chordal SLE_κ and study its relationships in different domains via so-called boundary perturbation properties. Then, in Section 2.4, we give a crucial coupling result for SLEs in different domains. This coupling is needed in the proof of Theorem 1.2 in Section 3.

2.1 Brownian Excursions and Brownian Loop Measure

We call a polygon $(\Omega; x, y)$ with two marked points a *Dobrushin domain*. Also, if $U \subset \Omega$ is a simply connected subdomain that agrees with Ω in neighborhoods of x and y, we say that U is a *Dobrushin subdomain* of Ω. For a Dobrushin domain $(\Omega; x, y)$, the *Brownian excursion measure* $\nu(\Omega; (yx))$ is a conformally invariant measure on Brownian excursions in Ω with its two endpoints in the arc (yx) — see [LW04, Section 3] for definitions. It is a σ-finite infinite measure, with the following restriction property: for any Dobrushin subdomain $U \subset \Omega$, we have

$$\nu(\Omega; (yx))[\,\cdot\,\mathbb{1}_{\{e \subset U\}}] = \nu(U; (yx))[\,\cdot\,]. \tag{2.1}$$

For $\xi \geq 0$, we call a Poisson point process with intensity $\xi\nu(\Omega; (yx))$ a *Brownian excursion soup.*

Suppose that x and y lie on analytic boundary segments of Ω. Then, the *boundary Poisson kernel* $H_\Omega(x, y)$ is a conformally invariant function which in the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \colon \Im z > 0\}$ with $x, y \in \mathbb{R}$ is given by

$$H_\mathbb{H}(x, y) = |y - x|^{-2} \tag{2.2}$$

(we do not include π^{-1} here), and in Ω it is defined via conformal covariance: we set

$$H_\Omega(x, y) = \varphi'(x)\varphi'(y) H_{\varphi(\Omega)}(\varphi(x), \varphi(y)) \tag{2.3}$$

for any conformal map $\varphi \colon \Omega \to \varphi(\Omega)$.

Lemma 2.1. *Let* $(\Omega; x, y)$ *be a Dobrushin domain with* x, y *on analytic boundary segments. Let* $U, V \subset \Omega$ *be two Dobrushin subdomains that agree with* Ω *in a neighborhood of the arc* (yx)*. Then we have*

$$H_\Omega(x, y) \; \geq H_U(x, y) \tag{2.4}$$

$$H_\Omega(x, y) \times H_{U \cap V}(x, y) \; \geq H_U(x, y) \times H_V(x, y). \tag{2.5}$$

Proof. The inequality (2.4) follows from (2.3). To prove (2.5), let $\mathcal{P}$ be a Brownian excursion soup with intensity $\nu(\Omega; (yx))$. Then we have

$$\mathbb{P}[e \subset U \ \forall \ e \in \mathcal{P}] = \frac{H_U(x,y)}{H_\Omega(x,y)}. \tag{2.6}$$

Now, denote by $\mathcal{P}_V$ the collection of excursions in $\mathcal{P}$ that are contained in V. By (2.1), we know that $\mathcal{P}_V$ is a Brownian excursion soup with intensity $\nu(V; (yx))$. Equation (2.5) now follows from

$$\frac{H_{U\cap V}(x,y)}{H_V(x,y)} = \mathbb{P}[e \subset U \ \forall \ e \in \mathcal{P}_V] \geq \mathbb{P}[e \subset U \ \forall \ e \in \mathcal{P}] = \frac{H_U(x,y)}{H_\Omega(x,y)}. \qquad \square$$

The *Brownian loop measure* $\mu(\Omega)$ is a conformally invariant measure on unrooted Brownian loops in Ω — see, e.g., [LW04, Sections 3 and 4] for the definition. It is a σ-finite infinite measure, which has the following restriction property: for any subdomain $U \subset \Omega$, we have

$$\mu(\Omega)[\,\cdot\, \mathbb{1}_{\{\ell \subset U\}}] = \mu(U)[\,\cdot\,].$$

For $\xi \geq 0$, we call a Poisson point process with intensity $\xi\mu(\Omega)$ a *Brownian loop soup*. This notion will be needed in Section 2.4.

Given two disjoint subsets $V_1, V_2 \subset \Omega$, we denote by $\mu(\Omega; V_1, V_2)$ the Brownian loop measure of loops in Ω that intersect both V_1 and V_2. In other words,

$$\mu(\Omega; V_1, V_2) = \mu\{\ell : \ \ell \subset \Omega, \ \ell \cap V_1 \neq \emptyset, \ \ell \cap V_2 \neq \emptyset\}.$$

If V_1, V_2 are at positive distance from each other, both of them are closed, and at least one of them is compact, then we have $0 \leq \mu(\Omega; V_1, V_2) < \infty$. Furthermore, the measure $\mu(\Omega; V_1, V_2)$ is conformally invariant: for any conformal transformation $\varphi \colon \Omega \to f(\Omega)$, we have $\mu(\varphi(\Omega); \varphi(V_1), \varphi(V_2)) = \mu(\Omega; V_1, V_2)$.

Also, for n disjoint subsets $V_1, \ldots, V_n$ of Ω, we denote by $\mu(\Omega; V_1, \ldots, V_n)$ the Brownian loop measure of loops in Ω that intersect all of $V_1, \ldots, V_n$. Again, provided that V_j are closed and at least one of them is compact, $\mu(\Omega; V_1, \ldots, V_n)$ is finite. This quantity will be needed in Section 3.

2.2 Loewner Chains and SLE

An $\mathbb{H}$-*hull* is a compact subset K of $\overline{\mathbb{H}}$ such that $\mathbb{H} \setminus K$ is simply connected. By Riemann's mapping theorem, for any hull K, there exists a unique conformal map g_K from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ such that $\lim_{z\to\infty} |g_K(z) - z| = 0$. Such a map g_K is called the conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ *normalized at* ∞. By standard estimates of conformal maps, the derivative of this map satisfies

$$0 < g_K'(x) \leq 1 \qquad \text{for all } x \in \mathbb{R} \setminus K. \tag{2.7}$$

In fact, this derivative can be viewed as the probability that the Brownian excursion in $\mathbb{H}$ from x to ∞ avoids the hull K — see [Vir03, LSW03].

Consider a family of conformal maps $(g_t, t \geq 0)$ which solve the Loewner equation: for each $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \qquad \text{and} \qquad g_0(z) = z,$$

where $(W_t, t \geq 0)$ is a real-valued continuous function, which we call the *driving function*.

Denote $K_t := \overline{\{z \in \mathbb{H} : T_z \leq t\}}$, where $T_z := \sup\{t \geq 0 : \inf_{s\in[0,t]} |g_s(z) - W_s| > 0\}$ is the *swallowing time* of the point z. Then, g_t is the unique conformal map from $H_t := \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ normalized at ∞. The collection of $\mathbb{H}$-hulls $(K_t, t \geq 0)$ associated with such maps is called a *Loewner chain*. We say that $(K_t, t \geq 0)$ is generated by the continuous curve $(\gamma(t), t \geq 0)$ if for any $t \geq 0$, the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ coincides with $H_t = \mathbb{H} \setminus K_t$.

In this article, we are concerned with particular hulls generated by curves. For $\kappa \geq 0$, the random Loewner chain $(K_t, t \geq 0)$ driven by $W_t = \sqrt{\kappa} B_t$, where $(B_t, t \geq 0)$ is a standard Brownian motion, is called the (chordal) *Schramm-Loewner Evolution* SLE_κ in $\mathbb{H}$ from 0 to ∞. S. Rohde and O. Schramm proved in [RS05] that this Loewner chain is almost surely generated by a continuous transient curve γ, with $\lim_{t\to\infty} |\gamma(t)| = \infty$. This random curve exhibits the following phase transitions in the parameter κ: when $\kappa \in [0, 4]$, it is a simple curve; whereas when $\kappa > 4$, it has self-touchings, being space-filling if $\kappa \geq 8$. The law of the SLE_κ curve is a probability measure on the space $X_0(\mathbb{H}; 0, \infty)$, and we denote it by $\mathbb{P}(\mathbb{H}; 0, \infty)$.

By conformal invariance, we can define the SLE_κ probability measure $\mathbb{P}(\Omega; x, y)$ in any simply connected domain Ω with two marked boundary points $x, y \in \partial\Omega$ (around which $\partial\Omega$ is locally connected) via pushforward of a conformal map: if $\gamma \sim \mathbb{P}(\mathbb{H}; 0, \infty)$, then we have $\varphi(\gamma) \sim \mathbb{P}(\Omega; x, y)$, where $\varphi \colon \mathbb{H} \to \Omega$ is any conformal map such that $\varphi(0) = x$ and $\varphi(\infty) = y$. In fact, by the results of O. Schramm [Sch00], the $(\mathrm{SLE}_\kappa)_{\kappa\geq 0}$ are the only probability measures on curves $\gamma \in X_0(\Omega; x, y)$ satisfying conformal invariance and the following domain Markov property: given an initial segment $\gamma[0, \tau]$ of the SLE_κ curve $\gamma \sim \mathbb{P}(\Omega; x, y)$ up to a stopping time τ, the conditional law of the remaining piece $\gamma[\tau, \infty)$ is the law $\mathbb{P}(\Omega \setminus K_\tau; \gamma(\tau), y)$ of the SLE_κ in the complement of the hull K_τ of the initial segment, from the tip $\gamma(\tau)$ to y.

2.3 Boundary Perturbation of SLE

The chordal SLE_κ curve $\gamma \sim \mathbb{P}(\Omega; x, y)$ has a natural boundary perturbation property, where its law in a Dobrushin subdomain of Ω is given by weighting by a factor involving the Brownian loop measure and the boundary Poisson kernel. More precisely, when $\kappa \in (0, 4]$, the SLE_κ is a simple curve only touching the boundary at its endpoints, and its law in the subdomain is absolutely continuous with respect to its law in Ω, as we state in the next Lemma 2.2. However, for $\kappa > 4$, we cannot have such absolute continuity since the SLE_κ has a positive chance to hit the boundary of Ω. Nevertheless, in Lemma 2.3 we show that if we restrict the two processes in a smaller domain, then we retain the absolute continuity for $\kappa \in (4, 8)$.

Throughout this article, we use the following real parameters:

$$h = \frac{6-\kappa}{2\kappa} \qquad \text{and} \qquad c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}. \tag{2.8}$$

Lemma 2.2. *Let $\kappa \in (0, 4]$. Let $(\Omega; x, y)$ be a Dobrushin domain and $U \subset \Omega$ a Dobrushin subdomain. Then, the SLE_κ in U connecting x and y is absolutely continuous with respect to the SLE_κ in Ω connecting x and y, with Radon-Nikodym derivative given by*

$$\frac{\mathrm{d}\mathbb{P}(U; x, y)}{\mathrm{d}\mathbb{P}(\Omega; x, y)}(\gamma) = \left(\frac{H_\Omega(x, y)}{H_U(x, y)}\right)^h \mathbb{1}_{\{\gamma\subset U\}} \exp(c\mu(\Omega; \gamma, \Omega \setminus U)).$$

Proof. See [LSW03, Section 5] and [KL07, Proposition 3.1]. □

The next lemma is a consequence of results in [LSW03, LW04]. We briefly summarize the proof.

Lemma 2.3. *Let $\kappa \in (4, 8)$. Let $(\Omega; x, y)$ be a Dobrushin domain. Let $\Omega^L \subset U \subset \Omega$ be Dobrushin subdomains such that Ω^L and Ω agree in a neighborhood of the arc (yx) and $\mathrm{dist}(\Omega^L, \Omega \setminus U) > 0$. Then, we have*

$$\mathbb{1}_{\{\gamma\subset\Omega^L\}} \frac{\mathrm{d}\mathbb{P}(U; x, y)}{\mathrm{d}\mathbb{P}(\Omega; x, y)}(\gamma) = \left(\frac{H_\Omega(x, y)}{H_U(x, y)}\right)^h \mathbb{1}_{\{\gamma\subset\Omega^L\}} \exp(c\mu(\Omega; \gamma, \Omega \setminus U)).$$

Proof. By conformal invariance, we may assume that $(\Omega; x, y) = (\mathbb{H}; 0, \infty)$. Let $\gamma \sim \mathbb{P}(\mathbb{H}; 0, \infty)$, let $(W_t, t \geq 0)$ be its driving function, and $(g_t, t \geq 0)$ the corresponding conformal maps. Let φ be the conformal map from U onto $\mathbb{H}$ normalized at ∞. On the event $\{\gamma \subset \Omega^L\}$, define T to be the first time that γ disconnects $\mathbb{H} \setminus U$ from ∞.

Denote by K_t the hull of $\gamma[0,t]$. For $t < T$, let $\tilde{g}_t$ be the conformal map from $\mathbb{H} \setminus \varphi(K_t)$ onto $\mathbb{H}$ normalized at ∞, and let φ_t be the conformal map from $g_t(U \setminus K_t)$ onto $\mathbb{H}$ normalized at ∞. Then we have $\tilde{g}_t \circ \varphi = \varphi_t \circ g_t$. Now, define for $t < T$,

$$M_t := \varphi_t'(W_t)^h \exp\left(-c \int_0^t \frac{S\varphi_s(W_s)}{6} \mathrm{d}s\right),$$

where Sf is the Schwarzian derivative[1]. It was proved in [LSW03, Proposition 5.3] that M_t is a local martingale. Furthermore, using Itô's formula, one can show that the law of γ weighted by M_t is $\mathbb{P}(U;0,\infty)$ up to time t. Also, it follows from [Law05, Proposition 5.22] (see also [LW04, Section 7]) that

$$-\int_0^t \frac{S\varphi_s(W_s)}{6} \mathrm{d}s = \mu(\mathbb{H};\gamma[0,t],\mathbb{H}\setminus U).$$

Now, on the event $\{\gamma \subset \Omega^L\}$, there exists $\epsilon = \epsilon(\mathbb{H},\Omega^L,U) > 0$ such that for $t < T$, we have $\epsilon \leq \varphi_t'(W_t) \leq 1$. When $\kappa \in (4,6]$, we have $h \geq 0$ and $c \geq 0$, and thus, on the event $\{\gamma \subset \Omega^L\}$, we have $M_t \leq \exp(c\mu(\mathbb{H};\Omega^L,\mathbb{H}\setminus U))$. When $\kappa \in (6,8)$, we have $h \leq 0$ and $c \leq 0$ and in this case, $M_t \leq \epsilon^h$. In either case, $(M_t, t < T)$ is uniformly bounded on the event $\{\gamma \subset \Omega^L\}$, and as $t \to T$, we have almost surely $\varphi_t'(W_t) \to 1$ and thus,

$$M_t \;\to\; M_T := \exp(c\mu(\mathbb{H};\gamma[0,T],\mathbb{H}\setminus U)).$$

The assertion follows taking into account that $M_0 = \varphi'(0)^h$ and recalling (2.3). □

2.4 A Crucial Coupling Result for SLEs

We finish this preliminary section with a result from [WW13], which says that we can construct SLEs using the Brownian loop soup and the Brownian excursion soup. This gives us a coupling of SLEs in two Dobrushin domains $U \subset \Omega$, which will be crucial in our proof of Theorem 1.2.

Let $(\Omega;x,y)$ be a Dobrushin domain. Let $\mathcal{L}$ be a Brownian loop soup with intensity $c\mu(\Omega)$, and $\mathcal{P}$ a Brownian excursion soup with intensity $h\nu(\Omega;(yx))$, with $c = c(\kappa)$ and $h = h(\kappa)$ defined in (2.8) and $\kappa \in (8/3,4]$. We note that then we have $c \in (0,1]$ and $h \in [1/4,5/8)$.

We say that two loops ℓ and ℓ' in $\mathcal{L}$ are in the same cluster if there exists a finite chain of loops $\ell_0,\ldots,\ell_n$ in $\mathcal{L}$ such that $\ell_0 = \ell$, $\ell_n = \ell'$, and $\ell_j \cap \ell_{j-1} \neq \emptyset$ for $j \in \{1,\ldots,n\}$. We denote by $\overline{\mathcal{C}}$ the family of all closures of the loop-clusters and by Γ the family of all outer boundaries of the outermost elements of $\overline{\mathcal{C}}$. Then, Γ forms a collection of disjoint simple loops called the CLE_κ for $\kappa \in (8/3,4]$, see [SW12].

Finally, define γ_0 to be the right boundary of the union of all excursions $e \in \mathcal{P}$ and γ to be the boundary of the union of γ_0 and all loops in Γ that it intersects, as illustrated in Figure 2.1.

Lemma 2.4. *[WW13, Theorem 1.1]. Let $\kappa \in [8/3,4]$. Let $(\Omega;x,y)$ be a Dobrushin domain and define $\mathcal{L}$, $\mathcal{P}$, Γ, γ_0, and γ as above. Then, γ has the law of the* SLE_κ *in Ω connecting x and y.*

From Lemma 2.4, we see that we can couple SLE_κ in different domains in the following way. Let $(\Omega;x,y)$ be a Dobrushin domain and $U \subset \Omega$ a Dobrushin subdomain that agrees with Ω in a neighborhood of the arc (yx). Take $\mathcal{L}$, $\mathcal{P}$, Γ, γ_0, and γ as in the above lemma. Let $\mathcal{P}_U$ and $\mathcal{L}_U$ respectively be the collections of excursions in $\mathcal{P}$ and loops in $\mathcal{L}$ that are contained in U. Define η_0 to be the right boundary of the union of all excursions $e \in \mathcal{P}_U$, define Γ_U to be the collection of all outer boundaries of the outermost clusters of $\mathcal{L}_U$, and η to be the right boundary of the union of η_0 and all loops in Γ_U that it intersects.

Corollary 2.5. *Let $(\Omega;x,y)$ be a Dobrushin domain and $U \subset \Omega$ a Dobrushin subdomain that agrees with Ω in a neighborhood of the arc (yx). There exists a coupling (γ,η) of $\gamma \sim \mathbb{P}(\Omega;x,y)$ and $\eta \sim \mathbb{P}(U,x,y)$ such that, almost surely, η stays to the left of γ and*

$$\mathbb{P}[\eta = \gamma] = \mathbb{P}[\gamma \subset U].$$

[1]The Schwarzian derivative of an analytic function f is given by $Sf(z) := \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}$.

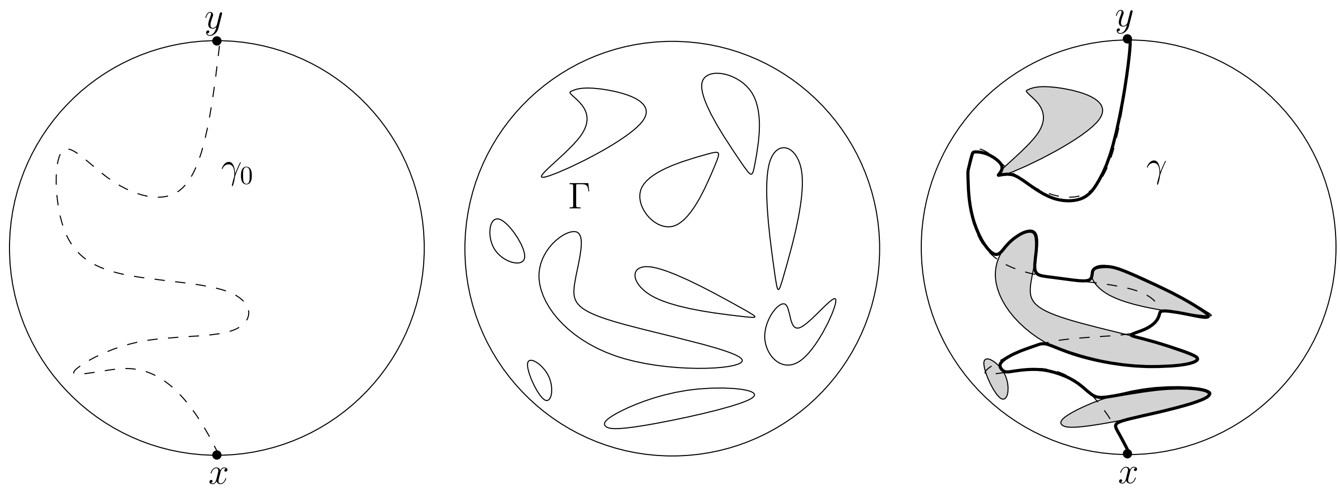

Figure 2.1: In the left panel, γ_0 is the right boundary of all Brownian excursions in $\mathcal{P}$. In the middle panel, Γ is the family of all outer boundaries of the outermost elements of the clusters of Brownian loops in $\mathcal{L}$. In the right panel, γ is the right boundary of the union of γ_0 and all loops in Γ that intersect γ_0. By [WW13, Theorem 1.1], we find that $\gamma \sim \mathbb{P}(\Omega; x, y)$.

Proof. Lemma 2.4 and the above paragraph give the sought coupling. □

In fact, the coupling (γ, η) of Corollary 2.5 is the coupling which maximizes the probability $\mathbb{P}[\eta = \gamma]$.

3 Global Multiple SLEs

Global N-SLEs associated to all link patterns $\alpha \in \mathrm{LP}_N$ and all $\kappa \in (0, 4]$ were constructed in the works [KL07, PW17]. This immediately gives the existence part of Theorem 1.2. In Section 3.1, we briefly recall the main idea of this construction. Then we prove the uniqueness part of Theorem 1.2 in Sections 3.2 and 3.3.

3.1 Construction of Global Multiple SLEs for $\kappa \leq 4$

Let $(\Omega; x_1, \ldots, x_{2N})$ be a polygon. For a link pattern $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$, we let $\mathbb{P}_\alpha$ denote the product measure of N independent chordal SLE_κ curves,

$$\mathbb{P}_\alpha := \bigotimes_{j=1}^{N} \mathbb{P}(\Omega; x_{a_j}, x_{b_j}),$$

and $\mathbb{E}_\alpha$ denote the expectation with respect to $\mathbb{P}_\alpha$. A global N-SLE_κ associated to α can be constructed as the probability measure $\mathbb{Q}_\alpha^\# = \mathbb{Q}_\alpha^\#(\Omega; x_1, \ldots, x_{2N})$ which is absolutely continuous with respect to $\mathbb{P}_\alpha$, with explicit Radon-Nikodym derivative given in Equation (3.2) below. This formula involves a combinatorial expression m_α of Brownian loop measures, obtained by an inclusion-exclusion procedure that depends on α. More precisely, for a configuration $(\eta_1, \ldots, \eta_N) \in X_0^\alpha(\Omega; x_1, \ldots, x_{2N})$, we define

$$m_\alpha(\Omega; \eta_1, \ldots, \eta_N) := \sum_{\text{c.c. } \mathcal{C} \text{ of } \Omega \setminus \{\eta_1, \ldots, \eta_N\}} m(\mathcal{C}), \tag{3.1}$$

where the sum is over all the connected components (c.c.) of the complement of the curves, and

$$m(\mathcal{C}) := \sum_{\substack{i_1, i_2 \in \mathcal{B}(\mathcal{C}),\\ i_1 \neq i_2}} \mu(\Omega; \eta_{i_1}, \eta_{i_2}) - \sum_{\substack{i_1, i_2, i_3 \in \mathcal{B}(\mathcal{C}),\\ i_1 \neq i_2 \neq i_3 \neq i_1}} \mu(\Omega; \eta_{i_1}, \eta_{i_2}, \eta_{i_3}) + \cdots + (-1)^p \mu(\Omega; \eta_{j_1}, \ldots, \eta_{j_p})$$

is a combinatorial expression associated to the c.c. $\mathcal{C}$, where

$$\mathcal{B}(\mathcal{C}) := \{j \in \{1, \ldots, N\} \colon \eta_j \subset \partial\mathcal{C}\} = \{j_1, \ldots, j_p\}$$

denotes the set of indices j for which the curve η_j is a part of the boundary of $\mathcal{C}$.

Now, we define the probability measure $\mathbb{Q}_\alpha^\#$ via

$$\begin{aligned} \frac{\mathrm{d}\mathbb{Q}_\alpha^\#}{\mathrm{d}\mathbb{P}_\alpha}(\eta_1, \ldots, \eta_N) &= \frac{R_\alpha(\Omega; \eta_1, \ldots, \eta_N)}{\mathbb{E}_\alpha[R_\alpha(\Omega; \eta_1, \ldots, \eta_N)]}, \\ \text{where} \qquad R_\alpha(\Omega; \eta_1, \ldots, \eta_N) &:= \mathbb{1}_{\{\eta_j \cap \eta_k = \emptyset \,\forall\, j \neq k\}} \exp(c m_\alpha(\Omega; \eta_1, \ldots, \eta_N)). \end{aligned} \tag{3.2}$$

By [PW17, Proposition 3.3], this measure satisfies the defining property of a global multiple SLE$_\kappa$. The expectation of R_α defines a conformally invariant and bounded function of the marked boundary points:

$$0 < f_\alpha(\Omega; x_1, \ldots, x_{2N}) := \mathbb{E}_\alpha[R_\alpha(\Omega; \eta_1, \ldots, \eta_N)] \leq 1. \tag{3.3}$$

If $(\Omega; x_1, \ldots, x_{2N})$ is a polygon and $U \subset \Omega$ a simply connected subdomain that agrees with Ω in neighborhoods of $x_1, \ldots, x_{2N}$, we say that U is a *sub-polygon* of Ω. When the marked points $x_1, \ldots, x_{2N}$ lie on analytic boundary segments of Ω, for all integers $N \geq 1$ and link patterns $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N$, we may define

$$\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N}) := f_\alpha(\Omega; x_1, \ldots, x_{2N}) \times \prod_{j=1}^N H_\Omega(x_{a_j}, x_{b_j})^h, \tag{3.4}$$

where H_Ω is the boundary Poisson kernel introduced in Section 2.1. Since $0 < f_\alpha \leq 1$, we see that

$$0 < \mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N}) \leq \prod_{j=1}^N H_\Omega(x_{a_j}, x_{b_j})^h. \tag{3.5}$$

The functions $\mathcal{Z}_\alpha$ are called *pure partition functions* for multiple SLEs. Explicit formulas for them have been obtained when $\kappa = 2$ [KKP17, Theorem 4.1] and $\kappa = 4$ [PW17, Theorem 1.5].

The multiple SLE probability measure $\mathbb{Q}_\alpha^\#$ has a useful boundary perturbation property. It serves as an analogue of Lemma 2.2 in our proof of Theorem 1.2.

Proposition 3.1. *[PW17, Proposition 3.4] Let $\kappa \in (0, 4]$. Let $(\Omega; x_1, \ldots, x_{2N})$ be a polygon and $U \subset \Omega$ a sub-polygon. Then, $\mathbb{Q}_\alpha^\#(U; x_1, \ldots, x_{2N})$ is absolutely continuous with respect to $\mathbb{Q}_\alpha^\#(\Omega; x_1, \ldots, x_{2N})$, with Radon-Nikodym derivative given by*

$$\frac{\mathrm{d}\mathbb{Q}_\alpha^\#(U; x_1, \ldots, x_{2N})}{\mathrm{d}\mathbb{Q}_\alpha^\#(\Omega; x_1, \ldots, x_{2N})}(\eta_1, \ldots, \eta_N) \;=\; \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_\alpha(U; x_1, \ldots, x_{2N})} \times \mathbb{1}_{\{\eta_j \subset U \,\forall\, j\}} \times \exp\Big(c\mu\Big(\Omega; \Omega \setminus U, \bigcup_{j=1}^N \eta_j\Big)\Big).$$

Moreover, when $\kappa \leq 8/3$ and $x_1, \ldots, x_{2N}$ lie on analytic boundary segments of Ω, we have

$$\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N}) \geq \mathcal{Z}_\alpha(U; x_1, \ldots, x_{2N}). \tag{3.6}$$

3.2 Uniqueness for a Pair of Commuting SLEs

Next, we prove that the global 2-SLE$_\kappa$ measures are unique. This result was first proved by J. Miller and S. Sheffield [MS16b, Theorem 4.1] by using a coupling of the SLEs with the Gaussian free field (GFF). We present another proof not using this coupling. Our proof also generalizes to the case of $N \geq 3$ commuting SLE curves, whereas couplings with the GFF seem not to be useful in that case.

In this section, we focus on polygons with $N = 2$. We call such a polygon $(\Omega; x_1, x_2, x_3, x_4)$ a *quad*. We also say that $U \subset \Omega$ is a *sub-quad* of Ω if U is a sub-polygon of Ω.

Because the two connectivities $\alpha \in \mathrm{LP}_2$ of the curves are obtained from each other by a cyclic change of labeling of the marked boundary points, we may without loss of generality consider global 2-SLEs associated to $\alpha = \{\{1,4\},\{2,3\}\}$. Hence, throughout this section, we consider pairs (η^L, η^R) of simple curves such that $\eta^L \in X_0(\Omega; x^L, y^L)$, and $\eta^R \in X_0(\Omega; x^R, y^R)$, and $\eta^L \cap \eta^R = \emptyset$. We denote the space of such pairs by $X_0(\Omega; x^L, x^R, y^R, y^L)$ Now, a probability measure supported on these pairs is a global 2-SLE_κ if the conditional law of η^L given η^R is that of the SLE_κ connecting x^L and y^L in the connected component of $\Omega \setminus \eta^R$ containing x^L and y^L on its boundary, and vice versa with R and L interchanged.

Proposition 3.2. *For any $\kappa \in (0,4]$, there exists a unique global 2-SLE_κ on $X_0(\Omega; x^L, x^R, y^R, y^L)$.*

We prove Proposition 3.2 in the end of this section, after some technical lemmas. The idea is to show that the global 2-SLE_κ is the unique stationary measure of a Markov chain which at each discrete time resamples one of the two curves according to its conditional law given the other one. We have already seen a construction of such a measure in the previous section, so we only need to prove that there exists at most one stationary measure. To this end, we use couplings of Markov chains — see e.g. [MT09] for a general background.

The next lemma is crucial in our proof. In this lemma, we prove that the chordal SLE_κ in Ω always has a uniformly positive probability of staying in a subdomain of Ω in the following sense.

Lemma 3.3. *Let $\kappa \in (0,4]$ and let $(\Omega; x, y)$ be a Dobrushin domain. Let $\Omega^L, U \subset \Omega$ be Dobrushin subdomains such that Ω^L, U, and Ω agree in a neighborhood of the arc (yx). Suppose $\eta \sim \mathbb{P}(U; x, y)$. Then, there exists a constant $\theta = \theta(\Omega, \Omega^L) > 0$ independent of U such that $\mathbb{P}[\eta \subset \Omega^L] \geq \theta$.*

Proof. We prove the lemma separately for $\kappa \in [8/3, 4]$ and $\kappa \in (0, 8/3]$.

When $\kappa \in [8/3, 4]$, we have $c \geq 0$. Suppose $\gamma \sim \mathbb{P}(\Omega; x, y)$ and denote by D_η (resp. D_γ) the connected component of $U \setminus \eta$ (resp. $\Omega \setminus \gamma$) with (yx) on its boundary. By Corollary 2.5, there exists a coupling of η and γ such that $D_\eta \subset D_\gamma$. Therefore, we have $\mathbb{P}[\eta \subset \Omega^L] \geq \mathbb{P}[\gamma \subset \Omega^L] > 0$. This gives the assertion for $\kappa \in [8/3, 4]$ with $\theta(\Omega, \Omega^L) = \mathbb{P}[\gamma \subset \Omega^L] > 0$.

Suppose then $\kappa \in (0, 8/3]$. Then we have $c \leq 0$. By Lemma 2.2, we have

$$\mathbb{P}[\eta \subset \Omega^L] = \left(\frac{H_\Omega(x,y)}{H_U(x,y)}\right)^h \mathbb{E}\big[\mathbb{1}_{\{\gamma \subset \Omega^L \cap U\}} \exp(c\mu(\Omega; \gamma, \Omega \setminus U))\big]. \tag{3.7}$$

Note that, on the event $\{\gamma \subset \Omega^L \cap U\}$, we have

$$\begin{aligned}\mu(\Omega; \gamma, \Omega \setminus (\Omega^L \cap U)) &= \mu(\Omega; \gamma, \Omega \setminus U) + \mu(\Omega; \gamma, \Omega \setminus \Omega^L) - \mu(\Omega; \gamma, \Omega \setminus \Omega^L, \Omega \setminus U) \\ &= \mu(\Omega; \gamma, \Omega \setminus U) + \mu(U; \gamma, U \setminus \Omega^L).\end{aligned} \tag{3.8}$$

Combining Equations (3.7) and (3.8) and using Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}\mathbb{P}[\eta \subset \Omega^L] &= \left(\frac{H_\Omega(x,y)}{H_U(x,y)}\right)^h \mathbb{E}\left[\mathbb{1}_{\{\gamma \subset \Omega^L \cap U\}} \exp(c\mu(\Omega; \gamma, \Omega \setminus U))\right] && \text{[by (3.7)]} \\ &\geq \left(\frac{H_\Omega(x,y)}{H_U(x,y)}\right)^h \mathbb{E}\left[\mathbb{1}_{\{\gamma \subset \Omega^L \cap U\}} \exp(c\mu(\Omega; \gamma, \Omega \setminus (\Omega^L \cap U)))\right] && \text{[by (3.8)]} \\ &= \left(\frac{H_{\Omega^L \cap U}(x,y)}{H_U(x,y)}\right)^h \geq \left(\frac{H_{\Omega^L}(x,y)}{H_\Omega(x,y)}\right)^h. && \text{[by Lemma 2.2 and (2.5)]}\end{aligned}$$

This gives the assertion for $\kappa \leq 8/3$ with $\theta(\Omega, \Omega^L) = (H_{\Omega^L}(x,y)/H_\Omega(x,y))^h > 0$. $\square$

Next, we prove that we can couple two SLEs in two Dobrushin subdomains of Ω in such a way that their realizations agree with a uniformly positive probability.

Lemma 3.4. *Let $\kappa \in (0,8)$ and let $(\Omega; x, y)$ be a Dobrushin domain. Let $\Omega^L \subset V \subset U, \tilde{U} \subset \Omega$ be Dobrushin subdomains such that Ω^L and Ω agree in a neighborhood of the arc (yx) and $\mathrm{dist}(\Omega^L, \Omega \setminus V) > 0$. Let $\eta \sim \mathbb{P}(U; x, y)$ and $\tilde{\eta} \sim \mathbb{P}(\tilde{U}; x, y)$. Then, there exists a coupling $(\eta, \tilde{\eta})$ such that $\mathbb{P}[\eta = \tilde{\eta} \subset \Omega^L] \geq \theta$, where the constant $\theta = \theta(\Omega, \Omega^L, V) > 0$ is independent of U and $\tilde{U}$.*

Proof. First, we show that there exists a constant $p_0 = p_0(\Omega, \Omega^L, V) > 0$, independent of U and $\tilde{U}$, such that $\mathbb{P}[\eta \subset \Omega^L] \geq p_0$. This is true for $\kappa \leq 4$ due to Lemma 3.3, so it remains to treat the case $\kappa \in (4, 8)$.

Let $\gamma \sim \mathbb{P}(\Omega; x, y)$. By Lemma 2.3, we have

$$\mathbb{P}[\eta \subset \Omega^L] = \left(\frac{H_\Omega(x,y)}{H_U(x,y)} \right)^h \mathbb{E}\left[\mathbb{1}_{\{\gamma \subset \Omega^L\}} \exp(c\mu(\Omega; \gamma, \Omega \setminus U)) \right].$$

When $\kappa \in (4, 6]$, we have $c \geq 0$ and $h \geq 0$. Combining with the inequality (2.4), we obtain

$$\mathbb{P}[\eta \subset \Omega^L] \geq \mathbb{P}[\gamma \subset \Omega^L].$$

On the other hand, when $\kappa \in (6, 8)$, we have $c \leq 0$ and $h \leq 0$. On the event $\{\gamma \subset \Omega^L\}$, we have $\mu(\Omega; \gamma, \Omega \setminus U) \leq \mu(\Omega; \Omega^L, \Omega \setminus V)$, so combining with (2.4), we obtain

$$\mathbb{P}[\eta \subset \Omega^L] \geq \left(\frac{H_\Omega(x,y)}{H_V(x,y)} \right)^h \exp(c\mu(\Omega; \Omega^L, \Omega \setminus V)) \mathbb{P}[\gamma \subset \Omega^L].$$

In either case, we have $\mathbb{P}[\eta \subset \Omega^L] \geq p_0$ with $p_0 = p_0(\Omega, \Omega^L, V) > 0$, independently of U and $\tilde{U}$.

Next, we consider the relation between $\tilde{\eta}$ and η. Using Lemmas 2.2 and 2.3, we see that the law of $\tilde{\eta}$ restricted to $\{\tilde{\eta} \subset \Omega^L\}$ is absolutely continuous with respect to the law of η restricted to $\{\eta \subset \Omega^L\}$, and the Radon-Nikodym derivative is given by

$$R(\eta) := \left(\frac{H_U(x,y)}{H_{\tilde{U}}(x,y)} \right)^h \mathbb{1}_{\{\eta \subset \Omega^L\}} \exp(c\mu(U; \eta, U \setminus \tilde{U}) - c\mu(\tilde{U}; \eta, \tilde{U} \setminus U)).$$

Now, Equation (2.4) shows that

$$\frac{H_V(x,y)}{H_\Omega(x,y)} \leq \frac{H_U(x,y)}{H_{\tilde{U}}(x,y)} \leq \frac{H_\Omega(x,y)}{H_V(x,y)}.$$

Also, because $\Omega^L \subset V \subset U, \tilde{U} \subset \Omega$, we see that on the event $\{\eta \subset \Omega^L\}$, we have

$$-\mu(\Omega; \Omega^L, \Omega \setminus V) \leq \mu(U; \eta, U \setminus \tilde{U}) - \mu(\tilde{U}; \eta, \tilde{U} \setminus U) \leq \mu(\Omega; \Omega^L, \Omega \setminus V).$$

These facts imply that $R(\eta) \geq \mathbb{1}_{\{\eta \subset \Omega^L\}} \epsilon$, where $\epsilon = \epsilon(\Omega, \Omega^L, V) > 0$ is independent of U and $\tilde{U}$.

Now, denote $\mathbb{P}[\eta \subset \Omega^L]$ by p. The total variation distance of the law of $\tilde{\eta}$ restricted to $\{\tilde{\eta} \subset \Omega^L\}$ and the law of η restricted to $\{\eta \subset \Omega^L\}$ is bounded from above by

$$\mathbb{E}[(1 - R(\eta))^+ \mathbb{1}_{\{\eta \subset \Omega^L\}}] \leq p - p\epsilon.$$

Thus, there exists a coupling $(\tilde{\eta}, \eta)$ such that $\mathbb{P}[\tilde{\eta} = \eta \subset \Omega^L] \geq p\epsilon$. From the first part of the proof, we see that $p \geq p_0(\Omega, \Omega^L, V)$. This proves the asserted result. □

It is important that the bounds in the technical Lemmas 3.3 and 3.4 are uniform over the domains U and $\tilde{U}$. In [MS16b, Lemma 4.2], the authors proved a seemingly similar result, but they only showed that there exists a coupling $(\eta, \tilde{\eta})$ such that $\mathbb{P}[\eta = \tilde{\eta}] > 0$, whereas in Lemma 3.4 we proved that $\mathbb{P}[\eta = \tilde{\eta}] \geq \theta$ with θ uniform over U and $\tilde{U}$. This is the key point in our proof of the uniqueness of Proposition 3.2.

Let us also emphasize that the assumption in Lemma 3.3 is $\Omega^L, U \subset \Omega$ and the assumption in Lemma 3.4 is $\Omega^L \subset U \subset \Omega$. Lemma 3.3 is the key point in the proof of the uniqueness, as it guarantees that there is a uniformly positive probability to couple two Markov chains for *any* initial values.

Remark 3.5. *It is worthwhile to discuss the optimal value of the constant θ in Lemmas 3.3 and 3.4. When $\kappa \in [8/3, 4]$, we know this optimal value exactly. Namely, from the proof of Lemma 3.3, we see that the optimal constant $\theta = \theta(\Omega, \Omega^L)$ equals $\mathbb{P}[\gamma \subset \Omega^L]$, where $\gamma \sim \mathbb{P}(\Omega; x, y)$. Also, in Lemma 3.4, if $\kappa \in [8/3, 4]$, then we can use the coupling of Corollary 2.5, which gives the optimal constant $\theta = \theta(\Omega, \Omega^L, V) = \mathbb{P}[\gamma \subset \Omega^L]$. In particular, this constant does not depend on V and Lemma 3.4 is true for all $\Omega^L \subset U, \tilde{U} \subset \Omega$.*

Proof of Proposition 3.2. The existence part is well-known, see [KL07] and Section 3.1 of the present article. Thus, we only need to prove the uniqueness.

By conformal invariance, we may take $\Omega = [0, \ell] \times [0, 1]$ and $x^L = (0, 0)$, $x^R = (\ell, 0)$, $y^R = (\ell, 1)$, $y^L = (0, 1)$. We define a Markov chain on curves $(\eta^L, \eta^R) \in X_0(\Omega; x^L, x^R, y^R, y^L)$ as follows. Given a configuration $(\eta^L_n, \eta^R_n) \in X_0(\Omega; x^L, x^R, y^R, y^L)$, we pick $i \in \{L, R\}$ uniformly and resample η^i_{n+1} according to the conditional law given the other curve. We will prove that this chain has at most one stationary measure.

Take two initial configurations (η^L_0, η^R_0) and $(\tilde{\eta}^L_0, \tilde{\eta}^R_0)$. We will show that there exists a constant $p_0 > 0$, independent of the initial configurations, so that the following is true: there exists a coupling of (η^L_4, η^R_4) and $(\tilde{\eta}^L_4, \tilde{\eta}^R_4)$ such that we have

$$\mathbb{P}[(\eta^L_4, \eta^R_4) = (\tilde{\eta}^L_4, \tilde{\eta}^R_4)] \geq p_0. \tag{3.9}$$

If (3.9) holds, then there exists a coupling of $(\eta^L_{4n}, \eta^R_{4n})$ and $(\tilde{\eta}^L_{4n}, \tilde{\eta}^R_{4n})$ for any $n \geq 1$ such that

$$\mathbb{P}[(\eta^L_{4n}, \eta^R_{4n}) \neq (\tilde{\eta}^L_{4n}, \tilde{\eta}^R_{4n})] \leq (1 - p_0)^n. \tag{3.10}$$

Bounding the total variation distance by this coupling, we see that it tends to zero as $n \to \infty$, so there exists at most one stationary measure. Hence, it is sufficient to prove (3.9).

Denote by $\Omega^L = [0, \ell/3] \times [0, 1]$ and $\Omega^R = [2\ell/3, \ell] \times [0, 1]$, and by $\theta_1 = \theta(\Omega, \Omega^L) = \theta(\Omega, \Omega^R)$ the constant obtained from Lemma 3.3. Given an initial configuration $(\eta^L_0, \eta^R_0) \in X_0(\Omega; x^L, x^R, y^R, y^L)$, we sample η^L_1 according to the conditional law and set $\eta^R_1 = \eta^R_0$. Then, we sample η^R_2 according to the conditional law and set $\eta^L_2 = \eta^L_1$. This operation has probability $1/4$. Knowing this sampling order, Lemma 3.3 gives

$$\mathbb{P}[\eta^L_2 \subset \Omega^L] \geq \theta_1 \qquad \text{and} \qquad \mathbb{P}[\eta^R_2 \subset \Omega^R \,|\, \eta^L_1] \geq \theta_1.$$

Thus, for any initial configurations, we have

$$\mathbb{P}\left[\eta^L_2 \subset \Omega^L,\ \eta^R_2 \subset \Omega^R\right] \geq \frac{1}{4}\theta_1^2. \tag{3.11}$$

Now, suppose we have two initial configurations (η^L_0, η^R_0) and $(\tilde{\eta}^L_0, \tilde{\eta}^R_0)$, and we sample (η^L_2, η^R_2) and $(\tilde{\eta}^L_2, \tilde{\eta}^R_2)$ independently. From (3.11), we see that

$$\mathbb{P}\left[\eta^L_2 \subset \Omega^L,\ \tilde{\eta}^L_2 \subset \Omega^L,\ \eta^R_2 \subset \Omega^R,\ \tilde{\eta}^R_2 \subset \Omega^R\right] \geq \frac{1}{16}\theta_1^4.$$

Given $(\eta^L_2, \eta^R_2, \tilde{\eta}^L_2, \tilde{\eta}^R_2)$, we resample η^L_3 and $\tilde{\eta}^L_3$ according to the conditional law and set $\eta^R_3 = \eta^R_2, \tilde{\eta}^R_3 = \tilde{\eta}^R_2$. Lemma 3.4 guarantees that there exists a coupling such that the probability of $\{\eta^L_3 = \tilde{\eta}^L_3 \subset \Omega^L\}$ is at least $\theta_2 > 0$, which does not depend on $(\eta^L_2, \eta^R_2, \tilde{\eta}^L_2, \tilde{\eta}^R_2)$ as long as $\{\eta^R_2, \tilde{\eta}^R_2 \subset \Omega^R\}$. Given $(\eta^L_3, \eta^R_3, \tilde{\eta}^L_3, \tilde{\eta}^R_3)$, we resample η^R_4 and $\tilde{\eta}^R_4$ according to the conditional law and set $\eta^L_4 = \eta^L_3, \tilde{\eta}^L_4 = \tilde{\eta}^L_3$. Similarly, there exists a coupling such that the probability of $\{\eta^R_4 = \tilde{\eta}^R_4 \subset \Omega^R\}$ is at least θ_2 as long as $\{\eta^L_3, \tilde{\eta}^L_3 \subset \Omega^L\}$. Therefore, there exists a coupling of (η^L_4, η^R_4) and $(\tilde{\eta}^L_4, \tilde{\eta}^R_4)$ such that

$$\mathbb{P}[(\eta^L_4, \eta^R_4) = (\tilde{\eta}^L_4, \tilde{\eta}^R_4)] \geq \frac{1}{64}\theta_1^4\theta_2^2.$$

This implies (3.9) with $p_0 = \frac{1}{64}\theta_1^4\theta_2^2$, and completes the proof. $\square$

The above proof works also when the conditional laws of η^R and η^L are variants of the chordal SLE_κ. In particular, we use this argument for certain SLE variants in the proof of Theorem 1.2.

Corollary 3.6. *Let $\kappa \in (0,4]$. For any $\alpha \in \mathrm{LP}_2$, there exists a unique global 2-SLE_κ associated to α.*

Proof. The two connectivities $\alpha \in \mathrm{LP}_2$ of the curves are obtained from each other by a cyclic change of labeling of the marked boundary points x_1, x_2, x_3, x_4. Thus, the assertion follows from Proposition 3.2. $\square$

3.3 Uniqueness: General Case

In this section, we generalize our uniqueness proof for the global 2-SLE_κ from the previous section to any number $N \geq 3$ of curves, to complete the proof of Theorem 1.2. Recall that, for $\alpha \in \mathrm{LP}_N$, we denote by $\mathbb{Q}_\alpha^\#(\Omega; x_1, \ldots, x_{2N})$ the global N-SLE_κ probability measures constructed in Section 3.

We begin by generalizing Lemma 3.3. By symmetry, we may assume that $\{1,2\} \in \alpha$ and denote $\hat{\alpha} = \alpha / \{1,2\}$.

Lemma 3.7. *Let $\kappa \in (0,4]$. Let $(\Omega; x_1, \ldots, x_{2N})$ be a polygon and $\Omega^L, U \subset \Omega$ sub-polygons such that Ω^L, U, and Ω agree in a neighborhood of the arc $(x_2\, x_1)$. Let $(\eta_1, \ldots, \eta_N)$ be any global N-SLE_κ in $(U; x_1, \ldots, x_{2N})$ such that η_1 is the curve connecting x_1 and x_2. Then, there exists a constant $\theta = \theta(\Omega, \Omega^L) > 0$, independent of U, such that $\mathbb{P}[\eta_1 \subset \Omega^L] \geq \theta$.*

Proof. Denote by $\hat{U}_1$ the connected component of $U \setminus \bigcup_{j=2}^N \eta_j$ with x_1 and x_2 on its boundary. Then, the conditional law of η_1 given $\hat{U}_1$ is the chordal SLE_κ in $\hat{U}_1$ connecting x_1 and x_2. By Lemma 3.3, we have $\mathbb{P}[\eta_1 \subset \Omega^L \,|\, \hat{U}_1] \geq \theta(\Omega, \Omega^L)$, independently of $\hat{U}_1$. Therefore, $\mathbb{P}[\eta_1 \subset \Omega^L] \geq \theta(\Omega, \Omega^L)$ as well. $\square$

To generalize Lemma 3.4, we use the following auxiliary result, which says that all of the curves have a positive probability to stay in a subdomain of Ω, uniformly with respect to a bigger subdomain.

Lemma 3.8. *Let $\kappa \in (0,4]$. Let $(\Omega; x_1, \ldots, x_{2N})$ be a polygon and $\Omega^L \subset U \subset \Omega$ sub-polygons. Let $(\eta_1, \ldots, \eta_N) \sim \mathbb{Q}_\alpha^\#(U; x_1, \ldots, x_{2N})$. Then, there exists a constant $\theta = \theta(\Omega, \Omega^L) > 0$, independent of U, such that $\mathbb{P}[\eta_j \subset \Omega^L \; \forall \, j] \geq \theta$.*

Proof. We prove the lemma separately for $\kappa \in [8/3, 4]$ and $\kappa \in (0, 8/3]$.

Assume first that $\kappa \in (0, 8/3]$. Let $(\gamma_1^L, \ldots, \gamma_N^L) \sim \mathbb{Q}_\alpha^\#(\Omega^L; x_1, \ldots, x_{2N})$. By Proposition 3.1, we have

$$\mathbb{P}[\eta_j \subset \Omega^L \; \forall \, j] = \frac{\mathcal{Z}_\alpha(\Omega^L; x_1, \ldots, x_{2N})}{\mathcal{Z}_\alpha(U; x_1, \ldots, x_{2N})} \, \mathbb{E}\Big[\exp\big(- c\mu\big(U; U \setminus \Omega^L, \bigcup_{j=1}^N \gamma_j^L\big)\big)\Big].$$

Since $\kappa \leq 8/3$, we have $c \leq 0$. Combining with (3.6), we obtain

$$\mathbb{P}[\eta_j \subset \Omega^L \; \forall \, j] \geq \frac{\mathcal{Z}_\alpha(\Omega^L; x_1, \ldots, x_{2N})}{\mathcal{Z}_\alpha(U; x_1, \ldots, x_{2N})} \geq \frac{\mathcal{Z}_\alpha(\Omega^L; x_1, \ldots, x_{2N})}{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})} > 0,$$

where the lower bound is independent of U, as claimed.

Assume next that $\kappa \in [8/3, 4]$. Let $(\gamma_1, \ldots, \gamma_N) \sim \mathbb{Q}_\alpha^\#(\Omega; x_1, \ldots, x_{2N})$. By Proposition 3.1, we have

$$\mathbb{P}[\eta_j \subset \Omega^L \; \forall \, j] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_\alpha(U; x_1, \ldots, x_{2N})} \, \mathbb{E}\Big[\mathbb{1}_{\{\gamma_j \subset U \; \forall \, j\}} \exp\big(c\mu\big(\Omega; \Omega \setminus U, \bigcup_{j=1}^N \gamma_j\big)\big)\Big].$$

Since $\kappa \in [8/3, 4]$, we have $c \geq 0$, so we get

$$\begin{aligned}
\mathbb{P}[\eta_j \subset \Omega^L \; \forall \, j] &\geq \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_\alpha(U; x_1, \ldots, x_{2N})} \, \mathbb{P}[\gamma_j \subset \Omega^L \; \forall \, j] \\
&\geq \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\prod_{j=1}^N H_U(x_{a_j}, x_{b_j})^h} \, \mathbb{P}[\gamma_j \subset \Omega^L \; \forall \, j] && \text{[by (3.5)]} \\
&\geq \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\prod_{j=1}^N H_\Omega(x_{a_j}, x_{b_j})^h} \, \mathbb{P}[\gamma_j \subset \Omega^L \; \forall \, j] > 0. && \text{[by (2.4)]}
\end{aligned}$$

This gives the assertion for $\kappa \in [8/3, 4]$ and finishes the proof. $\square$

Now, we prove an analogue of Lemma 3.4 for $\kappa \leq 4$.

Lemma 3.9. *Let $\kappa \in (0, 4]$. Let $(\Omega; x_1, \ldots, x_{2N})$ be a polygon and $\Omega^L \subset V \subset U, \tilde{U} \subset \Omega$ sub-polygons such that $\operatorname{dist}(\Omega^L, \Omega \setminus V) > 0$. Let $(\eta_1, \ldots, \eta_N) \sim \mathbb{Q}^{\#}_{\alpha}(U; x_1, \ldots, x_{2N})$ and $(\tilde{\eta}_1, \ldots, \tilde{\eta}_N) \sim \mathbb{Q}^{\#}_{\alpha}(\tilde{U}; x_1, \ldots, x_{2N})$. Then, there exists a coupling of $(\eta_1, \ldots, \eta_N)$ and $(\tilde{\eta}_1, \ldots, \tilde{\eta}_N)$ such that $\mathbb{P}[\eta_j = \tilde{\eta}_j \subset \Omega^L \ \forall\, j] \geq \theta$, where the constant $\theta = \theta(\Omega, \Omega^L, V) > 0$ is independent of U and $\tilde{U}$.*

Proof. By Proposition 3.1, the law of $(\tilde{\eta}_1, \ldots, \tilde{\eta}_N)$ restricted to $\{\tilde{\eta}_j \subset \Omega^L \ \forall\, j\}$ is absolutely continuous with respect to the law of $(\eta_1, \ldots, \eta_N)$ restricted to $\{\eta_j \subset \Omega^L \ \forall\, j\}$, with Radon-Nikodym derivative

$$R(\eta_1, \ldots, \eta_N) = \frac{\mathcal{Z}_{\alpha}(U; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\alpha}(\tilde{U}; x_1, \ldots, x_{2N})} \mathbb{1}_{\{\eta_j \subset \Omega^L \ \forall\, j\}} \exp\Big(c\mu\big(U; U \setminus \Omega^L, \bigcup_{j=1}^{N} \eta_j\big) - c\mu\big(\tilde{U}; \tilde{U} \setminus \Omega^L, \bigcup_{j=1}^{N} \eta_j\big)\Big).$$

First, we find a positive lower bound for $R(\eta_1, \ldots, \eta_N)$, separately for $\kappa \in [8/3, 4]$ and $\kappa \in (0, 8/3]$.

Since $\Omega^L \subset V \subset U, \tilde{U} \subset \Omega$, on the event $\{\eta_j \subset \Omega^L \ \forall\, j\}$, we have

$$-\mu(\Omega; \Omega \setminus V, \Omega^L) \leq \mu\big(U; U \setminus \Omega^L, \bigcup_{j=1}^{N} \eta_j\big) - \mu\big(\tilde{U}; \tilde{U} \setminus \Omega^L, \bigcup_{j=1}^{N} \eta_j\big) \leq \mu(\Omega; \Omega \setminus V, \Omega^L).$$

When $\kappa \in (0, 8/3]$, we have $c \leq 0$. Combining with (3.6), on the event $\{\eta_j \subset \Omega^L \ \forall\, j\}$, we have

$$R(\eta_1, \ldots, \eta_N) \geq \frac{\mathcal{Z}_{\alpha}(\Omega^L; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\alpha}(\Omega; x_1, \ldots, x_{2N})} \exp(c\mu(\Omega; \Omega \setminus V, \Omega^L)) > 0. \tag{3.12}$$

On the other hand, when $\kappa \in [8/3, 4]$, we have $c \geq 0$. Now, on the event $\{\eta_j \subset \Omega^L \ \forall\, j\}$, we have

$$R(\eta_1, \ldots, \eta_N) \geq \frac{\mathcal{Z}_{\alpha}(U; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\alpha}(\tilde{U}; x_1, \ldots, x_{2N})} \exp(-c\mu(\Omega; \Omega \setminus V, \Omega^L)).$$

Using (3.5) and (2.4), we estimate the denominator as

$$\mathcal{Z}_{\alpha}(\tilde{U}; x_1, \ldots, x_{2N}) \leq \prod_{j=1}^{N} H_{\tilde{U}}(x_{a_j}, x_{b_j})^h \leq \prod_{j=1}^{N} H_{\Omega}(x_{a_j}, x_{b_j})^h, \tag{3.13}$$

and using (2.4), we estimate the numerator as

$$\mathcal{Z}_{\alpha}(U; x_1, \ldots, x_{2N}) = \prod_{j=1}^{N} H_U(x_{a_j}, x_{b_j})^h f_{\alpha}(U; x_1, \ldots, x_{2N}) \geq \prod_{j=1}^{N} H_{\Omega^L}(x_{a_j}, x_{b_j})^h f_{\alpha}(U; x_1, \ldots, x_{2N}).$$

Taking the infimum over all sub-polygons A such that $V \subset A \subset \Omega$, we have

$$f_{\alpha}(U; x_1, \ldots, x_{2N}) \geq := \inf_{A} f_{\alpha}(A; x_1, \ldots, x_{2N}) := \upsilon(\Omega, V).$$

We next show that this infimum is strictly positive. By conformal invariance of f_{α}, we may take $\Omega = \mathbb{H}$, and we have $f_{\alpha}(A; x_1, \ldots, x_{2N}) = f_{\alpha}(\mathbb{H}; \varphi_A(x_1), \ldots, \varphi_A(x_{2N})) > 0$ for any conformal map $\varphi_A \colon A \to \mathbb{H}$. Now, there exists a compact subset K of $\mathbb{R}^{2N}$ such that $(\varphi_A(x_1), \ldots, \varphi_A(x_{2N})) \in K$ for all A, so

$$\upsilon(\Omega, V) = \inf_{(y_1, \ldots, y_{2N}) \in K} f_{\alpha}(\mathbb{H}; y_1, \ldots, y_{2N}) > 0.$$

Thus, we have

$$\mathcal{Z}_\alpha(U; x_1, \ldots, x_{2N}) \geq \prod_{j=1}^{N} H_{\Omega^L}(x_{a_j}, x_{b_j})^h \upsilon(\Omega, V) > 0. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$R(\eta_1, \ldots, \eta_N) \geq \prod_{j=1}^{N} \left(\frac{H_{\Omega^L}(x_{a_j}, x_{b_j}}{H_\Omega(x_{a_j}, x_{b_j}} \right)^h \upsilon(\Omega, V) \exp(-c\mu(\Omega; \Omega \setminus V, \Omega^L)) > 0. \tag{3.15}$$

In both estimates (3.12) and (3.15), we have $R(\eta_1, \ldots, \eta_N) \geq \epsilon := \epsilon(\Omega, \Omega^L, V) > 0$, independently of U and $\tilde{U}$, as desired. This completes the first part of the proof.

Now, denote $\mathbb{P}[\eta_j \subset \Omega^L \; \forall \, j]$ by p. The total variation distance of the law of $(\tilde{\eta}_1, \ldots, \tilde{\eta}_N)$ restricted to $\{\tilde{\eta}_j \subset \Omega^L \; \forall \, j\}$ and the law of $(\eta_1, \ldots, \eta_N)$ restricted to $\{\eta_j \subset \Omega^L \; \forall \, j\}$ is bounded from above by

$$\mathbb{E}\Big[(1 - R(\eta_1, \ldots, \eta_N))^+ \mathbb{1}_{\{\eta_j \subset \Omega^L \; \forall \, j\}}\Big] \leq p(1 - \epsilon).$$

It follows from this observation that there exists a coupling of $(\eta_1, \ldots, \eta_N)$ and $(\tilde{\eta}_1, \ldots, \tilde{\eta}_N)$ such that $\mathbb{P}[\tilde{\eta}_j = \eta_j \subset \Omega^L \; \forall \, j] \geq p\epsilon$. Combining with Lemma 3.8, we obtain the asserted result. □

We are ready to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. The existence was proved in [KL07, Law09, PW17], and summarized in Section 3. To prove the uniqueness, we proceed by induction on $N \geq 2$. The case $N = 2$ is the content of Corollary 3.6, so we let $N \geq 3$ and assume that, for any link pattern $\beta = \{\{a_1, b_1\}, \ldots, \{a_{N-1}, b_{N-1}\}\} \in \mathrm{LP}_{N-1}$, there exists a unique global $(N-1)$-SLE_κ associated to β. For $1 \leq j \leq N-1$, we denote by $\mathbb{Q}_\beta^{\{a_j, b_j\}}(\Omega; x_1, \ldots, x_{2N-2})$ the marginal law of η_j in this global multiple SLE.

Now, let $\alpha \in \mathrm{LP}_N$ and suppose that $(\eta_1, \ldots, \eta_N) \in X_0^\alpha(\Omega; x_1, \ldots, x_{2N})$ has the law of a global N-SLE_κ associated to α. By symmetry, we may assume that $\{1, 2\}, \{k, k+1\} \in \alpha$ with $k \in \{3, 4, \ldots, 2N-1\}$. Denote by η^L (resp. η^R) the curve in the collection $\{\eta_1, \ldots, \eta_N\}$ that connects x_1 and x_2 (resp. x_k and x_{k+1}). It follows from the induction hypothesis that the conditional law of the rest $(N-2)$ curves given (η^L, η^R) is the unique global $(N-2)$-SLE_κ associated to $(\alpha/\{k, k+1\})/\{1, 2\}$ in the appropriate remaining domain. Thus, it is sufficient to prove the uniqueness of the joint law on the pair (η^L, η^R).

The induction hypothesis also implies that, given η^R (resp. η^L), the conditional law of the rest of the curves is the unique global $(N-1)$-SLE_κ associated to $\alpha/\{k, k+1\}$ (resp. $\alpha/\{1, 2\}$). Denote by D^L (resp. D^R) the connected component of $\Omega \setminus \eta^R$ (resp. $\Omega \setminus \eta^L$) with x_1 and x_2 (resp. x_k and x_{k+1}) on its boundary. Then, the conditional law of η^L given η^R is $\mathbb{Q}_{\alpha/\{k,k+1\}}^{\{1,2\}}(D^L; x_1, \ldots, x_{k-1}, x_{k+2}, \ldots, x_{2N})$ and similarly, the conditional law of η^R given η^L is $\mathbb{Q}_{\alpha/\{1,2\}}^{\{k-2,k-1\}}(D^R; x_3, \ldots, x_{2N})$.

Following the idea of the proof of Proposition 3.2, we consider Markov chains sampling η^L and η^R from these conditional laws. Replacing in the proof of Proposition 3.2 Lemma 3.3 by Lemma 3.7 (for $N-1$) and Lemma 3.4 by Lemma 3.9 (also for $N-1$), one can show that this Markov chain has at most one stationary measure. Thus, the law of the collection $(\eta_1, \ldots, \eta_N) \in X_0^\alpha(\Omega; x_1, \ldots, x_{2N})$ is unique. □

To conclude this section, we give the marginal law of a single curve in the global multiple SLE_κ. Recall that the pure partition functions $\mathcal{Z}_\alpha$ were defined in (3.4). We denote

$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) := \mathcal{Z}_\alpha(\mathbb{H}; x_1, \ldots, x_{2N}), \qquad \text{for } x_1 < \cdots < x_{2N}.$$

Lemma 3.10. *[PW17, Proposition 4.9]. Let $\kappa \in (0,4]$ and $\alpha \in \mathrm{LP}_N$. Assume that $\{j,k\} \in \alpha$. Let W_t be the solution to the following SDEs:*

$$\begin{aligned} \mathrm{d}W_t &= \sqrt{\kappa}\mathrm{d}B_t + \kappa \partial_j \log \mathcal{Z}_\alpha \left(V_t^1, \ldots, V_t^{j-1}, W_t, V_t^{j+1}, \ldots, V_t^{2N}\right) \mathrm{d}t, \qquad W_0 = x_j \\ \mathrm{d}V_t^i &= \frac{2\mathrm{d}t}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq j. \end{aligned} \tag{3.16}$$

Then, the Loewner chain driven by W_t is well-defined up to the swallowing time T_k of x_k. Moreover, it is almost surely generated by a continuous curve up to and including T_k. This curve has the same law as the one connecting x_j and x_k in the global multiple SLE_κ associated to α in the polygon $(\mathbb{H}; x_1, \ldots, x_{2N})$.

4 Multiple Interfaces in Ising and Random-Cluster Models

In this final section, we give examples of discrete models whose interfaces converge in the scaling limit to multiple SLEs. More precisely, we consider the critical Ising and the random-cluster models in the plane.

In the case of the critical Ising model with alternating boundary conditions, K. Izyurov proved in his article [Izy17] that any number N of interfaces converges to a multiple SLE process in a local sense. In the present article, we condition the interfaces to have a given connectivity pattern and prove the convergence of the interfaces as a whole global collection of curves, which we know by Theorem 1.2 to be given by the unique global N-SLE_3 measure. This is the content of Section 4.1, where we prove Proposition 1.3. We are also able to determine the marginal law of one curve in this scaling limit. The case of two curves was considered in [Wu17]; in this case, the marginal law is the so-called hypergeometric SLE.

In Section 4.2, we consider interfaces in the critical random-cluster model, also with alternating boundary conditions and fixing the connectivity pattern of the curves. We show in Proposition 4.7 that, given the convergence of a single interface, multiple interfaces also have a conformally invariant scaling limit, which is the unique global multiple SLE_κ with $\kappa \in (4,6]$. This range of the parameter κ is beyond the range $(0,4]$ where global multiple SLEs have been explicitly constructed. Thus, from the converge of these discrete interfaces we would in fact get the existence and uniqueness of the global multiple SLE_κ with $\kappa \in (4,6]$. Unfortunately, the convergence of a single interface in the random-cluster model towards the chordal SLE_κ has only been rigorously established for the case $\kappa = 16/3$ — the FK-Ising model. This is the case appearing in Proposition 1.4, which we prove in Section 4.3. The convergence of two interfaces of the FK-Ising model was also proved in [KS18], where the authors used the so-called holomorphic observable constructed in [CS12]. In contrast, our method gives the convergence for any given number of interfaces via a global approach. We prove the case of two interfaces in Lemma 4.9, where the main effort is to show the uniqueness of the limit, and we establish the general case in Proposition 4.7.

In [PW17, Sections 5 and 6], the authors discussed multiple level lines of the Gaussian free field with alternating boundary conditions. These level lines give rise to global multiple SLE_4 curves (with any connectivity pattern). In this particular case, the marginal law of one curve in the global multiple SLE_4 degenerates to a certain $\mathrm{SLE}_4(\rho)$ process. In general, however, the marginal laws of single curves in global multiple SLEs are not $\mathrm{SLE}_\kappa(\rho)$ processes but rather certain more general variants of the chordal SLE_κ. We refer to [PW17, Section 3] for more details.

Notation and terminology. We will use the following notions throughout. For notational simplicity, we consider the square lattice $\mathbb{Z}^2$. Two vertices v and w are said to be neighbors if their Euclidean distance equals one, and we then write $v \sim w$. For a finite subgraph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G})) \subset \mathbb{Z}^2$, we denote by $\partial\mathcal{G}$ the inner boundary of $\mathcal{G}$: that is, $\partial\mathcal{G} = \{v \in V(\mathcal{G}) : \exists w \notin V(\mathcal{G}) \text{ such that } \{v,w\} \in E(\mathbb{Z}^2)\}$.

In the case of the square lattice, the *dual lattice* $(\mathbb{Z}^2)^*$ is just a translated version of $\mathbb{Z}^2$. More precisely, $(\mathbb{Z}^2)^*$ is the dual graph of $\mathbb{Z}^2$: its vertex set is $(1/2,1/2) + \mathbb{Z}^2$ and its edges are given by all pairs (v_1, v_2) of vertices that are neighbors. The vertices and edges of $(\mathbb{Z}^2)^*$ are called dual-vertices and dual-edges. In particular, for each edge e of $\mathbb{Z}^2$, we associate a dual edge, denoted by e^*, that crosses e in the middle.

For a subgraph $\mathcal{G}$ of $\mathbb{Z}^2$, we define $\mathcal{G}^*$ to be the subgraph of $(\mathbb{Z}^2)^*$ with edge set $E(\mathcal{G}^*) = \{e^* : e \in E(\mathcal{G})\}$ and vertex set given by the endpoints of these dual-edges.

Finally, the *medial lattice* $(\mathbb{Z}^2)^\diamond$ is the graph with the centers of edges of $\mathbb{Z}^2$ as the vertex set, and edges given by all pairs of vertices that are neighbors. In the case of the square lattice, the medial lattice is a rotated and rescaled version of $\mathbb{Z}^2$. We identify the faces of $(\mathbb{Z}^2)^\diamond$ with the vertices of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$.

Suppose $\mathcal{G}$ is a finite connected subgraph of the (possibly translated, rotated, and rescaled) square lattice $\mathbb{Z}^2$ such that the complement of $\mathcal{G}$ is also connected (this means that $\mathcal{G}$ is simply connected). Then, we call a triple $(\mathcal{G}; v, w)$ with $v, w \in \partial\mathcal{G}$ *a discrete Dobrushin domain.* We note that the boundary $\partial\mathcal{G}$ is divided into two parts (vw) and (wv), that we call arcs. More generally, given boundary vertices $v_1, \ldots, v_{2N} \in \partial\mathcal{G}$, we call the $(2N+1)$-tuple $(\mathcal{G}; v_1, \ldots, v_{2N})$ *a discrete (topological) polygon.* In this case, the boundary $\partial\mathcal{G}$ is divided into $2N$ arcs. As an abuse of notation, we sometimes let $\mathcal{G}$ also denote the simply connected domain formed by all of the faces, edges, and vertices of $\mathcal{G}$.

In this article, we consider *scaling limits* of models on discrete lattices with mesh size tending to zero. We only consider the following square lattice approximations, even though the results discussed in this section hold in a more general setting as well [CS12]. For small $\delta > 0$, we let Ω^δ denote a finite subgraph of the rescaled square lattice $\delta\mathbb{Z}^2$. Like Ω^δ, we decorate its vertices and edges with the mesh size δ as a superscript. The definitions of the dual lattice $\Omega^\delta_* := (\Omega^\delta)^*$, the medial lattice $\Omega^\delta_\diamond := (\Omega^\delta)^\diamond$, and discrete Dobrushin domains and polygons obviously extend to this context.

Let $(\Omega; x_1, \ldots, x_{2N})$ be a bounded polygon and $(\Omega^\delta; x^\delta_1, \ldots, x^\delta_{2N})$ a sequence of discrete polygons. We say that $(\Omega^\delta; x^\delta_1, \ldots, x^\delta_{2N})$ converges to $(\Omega; x_1, \ldots, x_{2N})$ *in the Carathéodory sense* if there exist conformal maps f^δ (resp. f) from the unit disc $\mathbb{U} = \{z \in \mathbb{C} \colon |z| < 1\}$ to Ω^δ (resp. from $\mathbb{U}$ to Ω) such that $f^\delta \to f$ on any compact subset of $\mathbb{U}$, and for all $j \in \{1, 2, \ldots, 2N\}$, we have $\lim_{\delta\to 0} (f^\delta)^{-1}(x^\delta_j) = f^{-1}(x_j)$.

4.1 Ising Model

Let $\mathcal{G}$ denote a finite subgraph of $\mathbb{Z}^2$. The Ising model on $\mathcal{G}$ with free boundary condition is a random assignment $\sigma \in \{\ominus, \oplus\}^{V(\mathcal{G})}$ of spins $\sigma_v \in \{\ominus, \oplus\}$, where σ_v denotes the spin at the vertex v. The Hamiltonian is defined by

$$H^{\mathrm{free}}_{\mathcal{G}}(\sigma) = -\sum_{v\sim w} \sigma_v \sigma_w. \tag{4.1}$$

The probability measure of the Ising model is given by the Boltzmann measure with Hamiltonian $H^{\mathrm{free}}_{\mathcal{G}}$ and inverse-temperature $\beta > 0$:

$$\mu^{\mathrm{free}}_{\beta,\mathcal{G}}[\sigma] = \frac{\exp(-\beta H^{\mathrm{free}}_{\mathcal{G}}(\sigma))}{Z^{\mathrm{free}}_{\beta,\mathcal{G}}}, \quad \text{where} \quad Z^{\mathrm{free}}_{\beta,\mathcal{G}} = \sum_{\sigma} \exp(-\beta H^{\mathrm{free}}_{\mathcal{G}}(\sigma)). \tag{4.2}$$

Also, for $\tau \in \{\ominus, \oplus\}^{\mathbb{Z}^2}$, we define the Ising model with boundary conditions τ via the Hamiltonian

$$H^{\tau}_{\mathcal{G}}(\sigma) = -\sum_{\substack{v\sim w,\\ \{v,w\}\cap\mathcal{G}\neq\emptyset}} \sigma_v \sigma_w, \quad \text{where} \quad \sigma_v = \tau_v \text{ for all } v \notin \mathcal{G}. \tag{4.3}$$

In particular, if $(\mathcal{G}; v, w)$ is a discrete Dobrushin domain, we may consider the Ising model with the following *Dobrushin boundary conditions* (domain-wall boundary conditions): we set $\oplus$ along the arc (vw), and $\ominus$ along the complementary arc (wv). More generally, we will consider the alternating boundary conditions (1.1), where $\oplus$ and $\ominus$ alternate along the boundary as in Figure 1.1.

A crucial point in the proof of Proposition 1.3 below is the following domain Markov property. Let $\mathcal{G} \subset \mathcal{G}'$ be two finite subgraphs of $\mathbb{Z}^2$. Fix $\tau \in \{\ominus, \oplus\}^{\mathbb{Z}^2}$ and $\beta > 0$. Let X be a random variable, which is measurable with respect to the status of the vertices in the smaller graph $\mathcal{G}$. Then we have

$$\mu^{\tau}_{\beta,\mathcal{G}'}\big[X \,|\, \sigma_v = \tau_v \,\forall v \in \mathcal{G}' \setminus \mathcal{G}\big] = \mu^{\tau}_{\beta,\mathcal{G}}[X].$$

The Ising model exhibits an order-disorder phase transition at a certain critical temperature. Above this temperature, the configurations are disordered and below it, the configurations have large clusters of equal spins. At criticality, the configurations have a self-similar behavior, and indeed, the critical planar Ising model is conformally invariant in the scaling limit [CS12, CHI15, CDCH⁺14]. On the square lattice, the critical value of β is

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}).$$

Now, we consider the scaling limit of the Ising model at criticality. Let $(\Omega^\delta_*; x^\delta_*, y^\delta_*)$ be a sequence of discrete Dobrushin domains converging to the bounded Dobrushin domain $(\Omega; x, y)$ in the Carathéodory sense. Consider the critical Ising model on Ω^δ_* with Dobrushin boundary conditions. Let $x^\delta_\diamond$ and $y^\delta_\diamond$ be vertices on the medial lattice $\Omega^\delta_\diamond$ nearest to x^δ_* and y^δ_*. Then, we define the Ising *interface* as follows. It starts from $x^\delta_\diamond$, traverses on the primal lattice Ω^δ, and turns at every vertex of Ω^δ in such a way that it always has dual vertices with spin $\oplus$ on its left and spin $\ominus$ on its right. If there is an indetermination when arriving at a vertex (this may happen on the square lattice), it turns left. See also Figure 4.1 for an illustration.

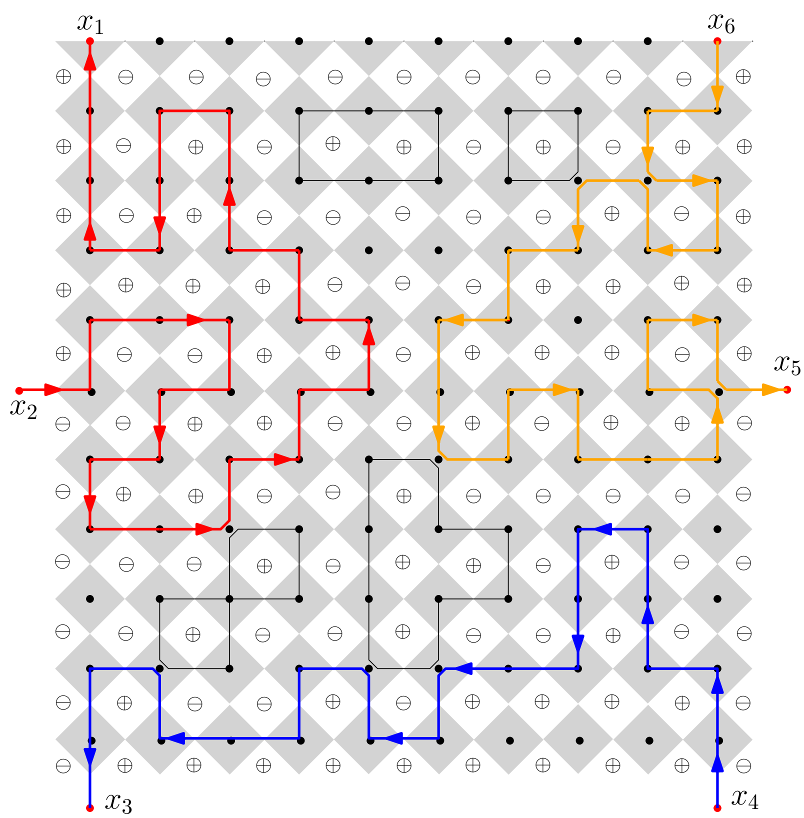

Figure 4.1: A spin configuration of the Ising model in a polygon with six marked points $x_1, \ldots, x_6$ on the boundary, with alternating boundary conditions. There are three interfaces starting from x_2, x_4, and x_6, illustrated in red, blue, and orange, respectively.

With this approximation scheme, we have the convergence of the interface to a conformally invariant scaling limit, the chordal SLE_κ with $\kappa = 3$.

Theorem 4.1. *[CDCH⁺14, Theorem 1]. Let $(\Omega^\delta_*; x^\delta_*, y^\delta_*)$ be a sequence of discrete Dobrushin domains converging to a Dobrushin domain $(\Omega; x, y)$ in the Carathéodory sense. Then, as $\delta \to 0$, the interface of the critical Ising model in $(\Omega^\delta_*, x^\delta_*, y^\delta_*)$ with Dobrushin boundary conditions converges weakly to the chordal* SLE_3 *in Ω connecting x and y.*

Using this result, we next prove that multiple interfaces also converge in the scaling limit to global multiple SLE_3 curves. Abusing notation, we write Ω^δ for $\Omega^\delta, (\Omega^\diamond)^\delta$, or $(\Omega^*)^\delta$, and x^δ for $x^\delta, (x^\diamond)^\delta$, or $(x^*)^\delta$. Let the polygons $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ converge to $(\Omega; x_1, \ldots, x_{2N})$ as $\delta \to 0$ in the Carathéodory sense. Consider the critical Ising model on Ω^δ with alternating boundary conditions (1.1). For $j \in \{1, \ldots, N\}$, let η_j^δ be the interface starting from x_{2j}^δ that separates $\oplus$ from $\ominus$. Then, the collection of interfaces $(\eta_1^\delta, \ldots, \eta_N^\delta)$ connects the boundary points $x_1^\delta, \ldots, x_{2N}^\delta$ forming a planar link pattern $\mathcal{A}^\delta \in \mathrm{LP}_N$. We consider the interfaces conditionally on forming a given connectivity $\mathcal{A}^\delta = \alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$. Our main goal is to prove Proposition 1.3:

Proposition 1.3. *Let $\alpha \in \mathrm{LP}_N$. Then, as $\delta \to 0$, conditionally on the event $\{\mathcal{A}^\delta = \alpha\}$, the law of the collection $(\eta_1^\delta, \ldots, \eta_N^\delta)$ of critical Ising interfaces converges weakly to the global N-SLE_3 associated to α. In particular, as $\delta \to 0$, the law of a single curve η_j^δ in this collection connecting two points x_j and x_k converges weakly to a conformal image of the Loewner chain with driving function given by Equation* (3.16) *in the end of Section 3, with $\kappa = 3$.*

Proof. Conditionally on $\{\mathcal{A}^\delta = \alpha\}$, we have $(\eta_1^\delta, \ldots, \eta_N^\delta) \in X_0^\alpha(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$. Following the same argument as in [Wu17, Section 5], we see that the collection of laws of the sequence $\{(\eta_1^\delta, \ldots, \eta_N^\delta)\}_{\delta > 0}$ is relatively compact; indeed, as proved in [KS17], the only ingredient needed to show the relative compactness is the Russo-Seymour-Welsh bound [CDCH16, Corollary 1.7]. Thus, there exist subsequential limits, and we may assume that, for some $\delta_n \to 0$, the sequence $(\eta_1^{\delta_n}, \ldots, \eta_N^{\delta_n})$ converges weakly to $(\eta_1, \ldots, \eta_N)$. For convenience, we couple them in the same probability space so that they converge almost surely. Also, to lighten the notation, we replace the superscripts δ_n by the superscript n here and in what follows. Finally, for each $j \in \{1, \ldots, N\}$, we let D_j^n denote the connected component of $\Omega^n \setminus \cup_{i \neq j} \eta_i^n$ having $x_{a_j}^n$ and $x_{b_j}^n$ on its boundary. The proof of the proposition consists in two lemmas, which we now sum up.

In Lemma 4.2, we show that the discrete Dobrushin domains $(D_j^n; x_{a_j}^n, x_{b_j}^n)$ converge almost surely to random Dobrushin domains in the Carathéodory sense. Notice that it is not clear that the limit of D_j^n is still simply connected, as the interfaces in the limit may touch the boundary, and they may have multiple points. The main point of the proof of Lemma 4.2 is therefore to rule out this behavior by arguments using Russo-Seymour-Welsh bounds and considering six-arm events. In particular, we show that the limit domain $(D_j; x_{a_j}, x_{b_j})$ is the simply connected subdomain D_j of $\Omega \setminus \cup_{i \neq j} \eta_i^n$ with x_{a_j}, x_{b_j} on its boundary. Lemma 4.2 also shows that $(\eta_1, \ldots, \eta_N) \in X_0^\alpha(\Omega; x_1, \ldots, x_{2N})$ almost surely.

Finally, we prove in Lemma 4.3 that the subsequential limit $(\eta_1, \ldots, \eta_N)$ must be a global multiple SLE_3. By Theorem 1.2, such an object is unique, thus being the unique subsequential limit. This gives the convergence of the sequence. The asserted marginal law of η_j follows from Lemma 3.10. □

Lemma 4.2. *In the setup of the proof of Proposition 1.3, as $n \to \infty$, for each $j \in \{1, \ldots, N\}$, the discrete Dobrushin domain $(D_j^n; x_{a_j}^n, x_{b_j}^n)$ converges almost surely to the Dobrushin domain $(D_j; x_{a_j}, x_{b_j})$ in the Carathéodory sense.*

Proof. First, in the case when x_{a_j} and x_{b_j} are neighbors, we can use the same argument as in [Wu17, Lemma 5.4]. We may assume that $b_j = a_j + 1$, and that the boundary conditions are $\oplus$ along $(x_{a_j}^n \, x_{b_j}^n)$ and $\ominus$ along $(x_{a_j-1}^n \, x_{a_j}^n)$ and $(x_{b_j}^n \, x_{b_j+1}^n)$. In this case, the boundary of the domain D_j^n contains the boundary arcs $(x_{a_j-1}^n \, x_{a_j}^n)$, $(x_{a_j}^n \, x_{b_j}^n)$, and $(x_{b_j}^n \, x_{b_j+1}^n)$, and some other parts which we denote by $\partial^L D_j^n$.

Denote by $\mathcal{C}_j^\ominus$ the event that there is a crossing of $\ominus$ in Ω^n connecting the boundary arcs $(x_{a_j-1}^n \, x_{a_j}^n)$ and $(x_{b_j}^n \, x_{b_j+1}^n)$. Note that $\{\mathcal{A}^n = \alpha\}$ implies this event $\mathcal{C}_j^\ominus$. Let d_j^n denote the extremal distance between $(x_{a_j}^n \, x_{b_j}^n)$ and $\partial^L D_j^n$ in D_j^n. By the Russo-Seymour-Welsh bound [CDCH16, Corollary 1.4], we know that the probability of $\mathcal{C}_j^\ominus$, given that d_j^n is small, is uniformly small: there exists a function $f \colon [0, \infty) \to [0, 1]$ such that $f(0) = 0$ and f is continuous at 0, and for any $u > 0$ and small enough $\delta_n > 0$, we have

$$\mathbb{P}\left[d_j^n \leq u \,|\, \mathcal{A}^n = \alpha\right] = \frac{\mathbb{P}\left[\{d_j^n \leq u\} \cap \{\mathcal{A}^n = \alpha\}\right]}{\mathbb{P}[\mathcal{A}^n = \alpha]} \leq \frac{\mathbb{P}\left[\{d_j^n \leq u\} \cap \mathcal{C}_j^\ominus\right]}{\mathbb{P}[\mathcal{A}^n = \alpha]} \leq \frac{f(u)}{\mathbb{P}[\mathcal{A}^n = \alpha]}.$$

This implies that the limit of d_j^n is almost surely strictly positive as $\delta_n \to 0$. Hence, by the Carathéodory kernel theorem [Pom92, Theorem 1.8], the random discrete Dobrushin domain $(D_j^n; x_{a_j}^n, x_{b_j}^n)$ converges almost surely to the random Dobrushin domain $(D_j; x_{a_j}, x_{b_j})$ in the Carathéodory sense.

Second, in the case when x_{a_j} and x_{b_j} are not neighbors, we need to argue that those parts of the boundary of D_j^n which are given by the interfaces also stay at a positive distance from each other, so that D_j^n stays simply connected in the limit $\delta_n \to 0$. For this, we use estimates of interior six-arm events.

For a vertex $v^n \in V(\Omega^n)$ and radii $r > r' > 0$, we let $\mathcal{E}^n(v^n; r, r')$ denote the event that the annulus $B(v^n, r) \setminus B(v^n, r')$ contains six disjoint monochromatic crossings and the pattern of these six crossings is alternating, $(\oplus \ominus \oplus \ominus \oplus \ominus)$. It was proved in [Wu18a, Theorem 1.2] that the probability of this event is

$$\mathbb{P}[\mathcal{E}^n(v^n; r, r')] = (r'/r)^{\alpha_6 + o(1)} \quad \text{as } \delta_n \to 0, \quad \text{where } \alpha_6 = 143/24.$$

In particular, there exist $C < \infty$ and $\aleph > 2$ so that, for $r' < r$ and for δ_n small enough, we have

$$\mathbb{P}[\mathcal{E}^n(v^n; r, r')] \leq C (r'/r)^{\aleph}.$$

Let us cover Ω^n with $N_{r'}$ balls of radius r' centered at vertices $\{v_j^n\}_{j \in \mathcal{N}_{r'}}$. Then, for each vertex $v^n \in V(\Omega^n)$, there exists a vertex $v_j^n \in \mathcal{N}_{r'}$ such that $|v^n - v_j^n| \leq r'$. Let $r \geq 6r'$. Then, the event $\mathcal{E}^n(v^n; r, r')$ implies the event $\mathcal{E}^n(v^n; r/2, 3r')$, so we have

$$\mathbb{P}\Bigg[\bigcup_{v^n \in V(\Omega^n)} \mathcal{E}^n(v^n; r, r')\Bigg] \leq \mathbb{P}\Bigg[\bigcup_{j \in \mathcal{N}_{r'}} \mathcal{E}^n(v_j^n; r/2, 3r')\Bigg] \leq 6^{\aleph} C N_{r'} (r'/r)^{\aleph}.$$

Because $\aleph > 2$ and $N_{r'}$ grows like $1/(r')^2$ as $r' \to 0$, we have

$$\lim_{r' \to 0} \limsup_{\delta_n \to 0} \mathbb{P}\Bigg[\bigcup_{v^n \in V(\Omega^n)} \mathcal{E}^n(v^n; r, r')\Bigg] = 0, \quad \text{for any } r > 0.$$

This shows that the collection $\{\eta_1, \ldots, \eta_N\}$ of curves cannot have triple points. Hence, we see that $(D_j^n; x_{a_j}^n, x_{b_j}^n)$ converges in the Carathéodory sense to $(D_j; x_{a_j}, x_{b_j})$ as $\delta_n \to 0$. □

We note that the proof of Lemma 4.2 also shows that $(\eta_1, \ldots, \eta_N) \in X_0^{\alpha}(\Omega; x_1, \ldots, x_{2N})$.

Lemma 4.3. *Still in the setup of the proof of Proposition 1.3, the limit $(\eta_1, \ldots, \eta_N)$ has the distribution of a global multiple* SLE_3.

Proof. We need to prove that, for each $j \in \{1, \ldots, N\}$, the conditional law of the random curve $X := \eta_j$ given the other random curves $Y := (\eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_N)$ is the appropriate chordal SLE_3. Denote

$$X^n := \eta_j^n \qquad \text{and} \qquad Y^n := (\eta_1^n, \ldots, \eta_{j-1}^n, \eta_{j+1}^n, \ldots, \eta_N^n).$$

By assumption, (X^n, Y^n) converges to (X, Y) in distribution as $\delta_n \to 0$. However, this does not automatically imply the convergence of the conditional distribution of X^n given Y^n to the conditional distribution of X given Y. In our case this is true, as we will now prove. (See also the discussion in [GW18, Section 5].)

Recall that we couple all $\{(X^n, Y^n)\}_{\delta_n \geq 0}$ in the same probability space so that they converge almost surely to (X, Y) as $\delta_n \to 0$. Now, given Y^n, the random curve X^n is an Ising interface with Dobrushin boundary conditions in the random Dobrushin domain $(D_j^n; x_{a_j}^n, x_{b_j}^n)$. By Lemma 4.2, almost surely, $(D_j^n; x_{a_j}^n, x_{b_j}^n)$ converges to the random Dobrushin domain $(D_j; x_{a_j}, x_{b_j})$ in the Carathéodory sense. Thus, almost surely, there exist conformal maps G^n (resp. G) from $\mathbb{U}$ onto D_j^n (resp. D_j) such that, as $\delta_n \to 0$, the maps G^n converge to G uniformly on compact subsets of $\mathbb{U}$, and we have $(G^n)^{-1}(x_{a_j}^n) \to G^{-1}(x_{a_j}) = 1$ and $(G^n)^{-1}(x_{b_j}^n) \to G^{-1}(x_{b_j}) = -1$. Furthermore, for each n, the map G^n is a measurable function of Y^n, and G is a measurable function of Y. We use the following two observations.

1. On the one hand, Theorem 4.1 shows that the law of $(G^n)^{-1}(X^n)$ converges to the chordal SLE_3 in $\mathbb{U}$ connecting the points 1 and -1.

2. On the other hand, we can show that $(G^n)^{-1}(X^n)$ converges to $G^{-1}(X)$. By assumption, (X^n, Y^n) converges to (X, Y) almost surely. Now, we send X^n (resp. X) conformally onto $\mathbb{H}$ and denote by W^n (resp. W) its driving function. It follows from the Russo-Seymour-Welsh bounds [CDCH16] for the critical Ising interfaces that they satisfy the so-called "Condition C2" in [KS17]. Hence, applying [KS17, Theorem 1.5 and Corollary 1.7] to $\{X^n\}_n$, we know that $W^n \to W$ locally uniformly. Also, applying [KS17, Theorem 1.5 and Corollary 1.7] to $\{(G^n)^{-1}(X^n)\}_n$, we know that this collection is tight (we remark that this fact is highly no-trivial and it was proved in [KS17]), and that, for any convergent subsequence $(G^{n_k})^{-1}(X^{n_k}) \to \tilde{\eta}$, the curve $\tilde{\eta}$ has a continuous driving function $\tilde{W}$ such that $W^n \to \tilde{W}$ locally uniformly. Combining these two facts, we see that $\tilde{W}$ coincides with W, so $\tilde{\eta}$ coincides with $G^{-1}(X)$. In particular, this is the only subsequential limit of the collection $\{(G^n)^{-1}(X^n)\}_n$, so $(G^n)^{-1}(X^n) \to G^{-1}(X)$.

Combining these two observations, we see that the law of $G^{-1}(X)$ is the chordal SLE_3 in $\mathbb{U}$ connecting 1 and -1. In particular, the law of $G^{-1}(X)$ is independent of Y with G a measurable function of Y. Hence, the conditional law of X given Y is the chordal SLE_3 in D_j connecting the points x_{a_j} and x_{b_j}. $\square$

4.2 Random-Cluster Model

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a finite subgraph of $\mathbb{Z}^2$. A *configuration* $\omega = (\omega_e \colon e \in E(\mathcal{G}))$ is an element of $\{0,1\}^{E(\mathcal{G})}$. If $\omega_e = 1$, the edge e is said to be *open*, and otherwise, e is said to be *closed*. The configuration ω can be seen as a subgraph of $\mathcal{G}$ with the same set of vertices $V(\mathcal{G})$, and whose edges are the open edges $\{e \in E(\mathcal{G}) \colon \omega_e = 1\}$. We denote by $o(\omega)$ (resp. $c(\omega)$) the number of open (resp. closed) edges of ω.

We are interested in the connectivity properties of the graph ω. The maximal connected components of ω are called *clusters*. Two vertices v and w are connected by ω inside $S \subset \mathbb{Z}^2$ if there exists a path of vertices $(v_i)_{0 \le i \le k}$ in S such that $v_0 = v$, $v_k = w$, and each edge $\{v_i, v_{i+1}\}$ is open in ω, for $0 \le i < k$.

We may also impose to our model various boundary conditions, which can be understood as encoding how the sites are connected outside $\mathcal{G}$. A *boundary condition* ξ is a partition $P_1 \sqcup \cdots \sqcup P_k$ of $\partial\mathcal{G}$. Two vertices are said to be *wired* in ξ if they belong to the same P_i, and *free* otherwise. We denote by ω^ξ the (quotient) graph obtained from the configuration ω by identifying the wired vertices together in ξ.

The probability measure $\phi^\xi_{p,q,\mathcal{G}}$ of the *random-cluster model* on $\mathcal{G}$ with edge-weight $p \in [0,1]$, cluster-weight $q > 0$, and boundary condition ξ, is defined by

$$\phi^\xi_{p,q,\Omega}[\omega] := \frac{p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^\xi)}}{Z^\xi_{p,q,\Omega}},$$

where $k(\omega^\xi)$ is the number of connected components of the graph ω^ξ, and $Z^\xi_{p,q,\Omega}$ is the normalizing constant to make $\phi^\xi_{p,q,\mathcal{G}}$ a probability measure. For $q = 1$, this model is simply Bernoulli bond percolation.

For a configuration ξ on $E(\mathbb{Z}^2) \setminus E(\mathcal{G})$, the boundary condition induced by ξ is defined as the partition $P_1 \sqcup \cdots \sqcup P_k$, where v and w belong to the same P_i if and only if there exists an open path in ξ connecting them. We identify the boundary condition induced by ξ with the configuration itself, and denote the measure of the random-cluster model with such boundary conditions by $\phi^\xi_{p,q,\mathcal{G}}$. As a direct consequence of these definitions, we have the following domain Markov property. Suppose that $\mathcal{G} \subset \mathcal{G}'$ are two finite subgraphs of $\mathbb{Z}^2$. Fix $p \in [0,1]$, $q > 0$, and a boundary condition ξ on $\partial\mathcal{G}'$. Let X be a random variable which is measurable with respect to the status of the edges in $\mathcal{G}$. Then we have

$$\phi^\xi_{p,q,\mathcal{G}'}\left[X \mid \omega_e = \psi_e, \text{for all } e \in E(\mathcal{G}') \setminus E(\mathcal{G})\right] = \phi^{\psi^\xi}_{p,q,\mathcal{G}}[X], \quad \text{for all } \psi \in \{0,1\}^{E(\mathcal{G}') \setminus E(\mathcal{G})}, \tag{4.4}$$

where ψ^ξ is the partition on $\partial\mathcal{G}$ obtained by wiring two vertices $v, w \in \partial\mathcal{G}$ if they are connected in ψ.

For $\omega, \omega' \in \{0,1\}^{E(\mathcal{G})}$, we denote by $\omega \le \omega'$ if $\omega_e \le \omega'_e$, for all $e \in E(\mathcal{G})$. An event $\mathcal{A}$ depending on the edges in $E(\mathcal{G})$ is said to be *increasing* if for any $\omega \in \mathcal{A}$, the inequality $\omega \le \omega'$ implies that $\omega' \in \mathcal{A}$. When $q \ge 1$, the following FKG inequality (positive association) holds. Fix $p \in [0,1], q \ge 1$, and a boundary condition ξ on $\partial\mathcal{G}$. Then, for any two increasing events $\mathcal{A}$ and $\mathcal{B}$, we have

$$\phi^{\xi}_{p,q,\mathcal{G}}[\mathcal{A} \cap \mathcal{B}] \ge \phi^{\xi}_{p,q,\mathcal{G}}[\mathcal{A}]\phi^{\xi}_{p,q,\mathcal{G}}[\mathcal{B}].$$

Consequently, for any boundary conditions $\xi \le \psi$ and for any increasing event $\mathcal{A}$, we have

$$\phi^{\xi}_{p,q,\mathcal{G}}[\mathcal{A}] \le \phi^{\psi}_{p,q,\mathcal{G}}[\mathcal{A}]. \tag{4.5}$$

A configuration ω on $\mathcal{G}$ can be uniquely associated to a dual configuration ω^* on the dual graph $\mathcal{G}^*$, defined by $\omega^*(e^*) = 1 - \omega(e)$ for all $e \in E(\mathcal{G})$. A dual-edge e^* is said to be *dual-open* if $\omega^*(e^*) = 1$ and *dual-closed* otherwise. A *dual-cluster* is a connected component of ω^*. We extend the notions of dual-open paths and connectivity events in the obvious way. Now, if ω is distributed according to $\phi^{\xi}_{p,q,\mathcal{G}}$, then ω^* is distributed according to $\phi^{\xi^*}_{p^*,q^*,\mathcal{G}^*}$, with

$$q^* = q \qquad \text{and} \qquad \frac{pp^*}{(1-p)(1-p^*)} = q.$$

Note that, at $p^* = p$, we have

$$p = p_c(q) := \frac{\sqrt{q}}{1+\sqrt{q}}.$$

For this critical case $p = p_c(q)$, we have the following generalized Russo-Symour-Welsh estimates. For a rectangle $R = [a,b] \times [c,d] \subset \mathbb{Z}^2$, we let $\mathcal{C}_h(R)$ denote the event that there exists an open path in R from $\{a\} \times [c,d]$ to $\{b\} \times [c,d]$.

We also denote by $\phi^0_{p,q,\mathcal{G}}$ the probability measure of the random-cluster model with *free boundary conditions*, where the partition ξ of $\partial\mathcal{G}$ consists of singletons only. We observe that in the sense of (4.5), this boundary condition is minimal.

Proposition 4.4. *[DCST17, Theorem 7]. Let $1 \le q < 4$ and $\epsilon > 0$, and denote by R_n the rectangle $[0, \epsilon n] \times [0, n]$. Then, there exists a constant $\theta(\epsilon) > 0$ such that we have*

$$\phi^0_{p_c(q),q,R_n}[\mathcal{C}_h(R_n)] \ge \theta(\epsilon) \qquad \text{for any } n \ge 1. \tag{4.6}$$

Proposition 4.4 plays an essential role in Section 4.3.

If $(\mathcal{G}; v, w)$ is a discrete Dobrushin domain, in the *Dobrushin boundary conditions*, all edges along the arc (wv) are open and all edges along (vw) are free. Then, for each vertex u of the medial graph $\mathcal{G}^\diamond$, there exists either an open edge of $\mathcal{G}$ or a dual-open edge of $\mathcal{G}^*$ passing through u. In addition, we can draw self-avoiding loops on $\mathcal{G}^\diamond$ as follows: a loop arriving at a vertex of the medial lattice always makes a turn of $\pm\pi/2$, so as not to cross the open or dual-open edges through this vertex. This loop-representation of the random-cluster model is illustrated in Figure 4.2. The loop representation contains loops and the self-avoiding path connecting two vertices $v^\diamond$ and $w^\diamond$ of the medial graph $\mathcal{G}^\diamond$ that are closest to v and w. This curve is called the *interface* (the exploration path) of the random-cluster model.

At the critical point $p = p_c(q)$, this interface is expected to converge to the chordal SLE_κ curve in the scaling limit, with κ specifically given by q. The convergence has been rigorously established for the special case of $q = 2$, also known as the FK-Ising model [CS12, CDCH$^+$14], in the topology of Section 1.2.

Conjecture 4.5. *[see, e.g., [Sch07]] Let $0 \le q \le 4$ and $p = p_c(q)$. Let $(\Omega^\delta; x^\delta, y^\delta)$ be a sequence of discrete Dobrushin domains converging to a Dobrushin domain $(\Omega; x, y)$ in the Carathéodory sense. Then, as $\delta \to 0$, the interface of the critical random-cluster model in $(\Omega^\delta; x^\delta, y^\delta)$ with cluster weight q and Dobrushin boundary conditions converges weakly to the chordal* SLE_κ *connecting x and y, with*

$$\kappa = \frac{4\pi}{\arccos(-\sqrt{q}/2)}. \tag{4.7}$$

Theorem 4.6. *[CDCH$^+$14, Theorem 2]. Conjecture 4.5 holds for $q = 2$ and $\kappa = 16/3$.*

4.3 Global Multiple SLEs with $\kappa \in (4, 6]$

In Section 4.2, we discussed the convergence of one interface in the critical random-cluster model with Dobrushin boundary conditions. In the present section, we consider the convergence of the collection of interfaces in the following setup. Let $N \geq 2$ and let $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ be a discrete polygon. Consider the critical random-cluster model in Ω^δ with alternating boundary conditions (1.3).

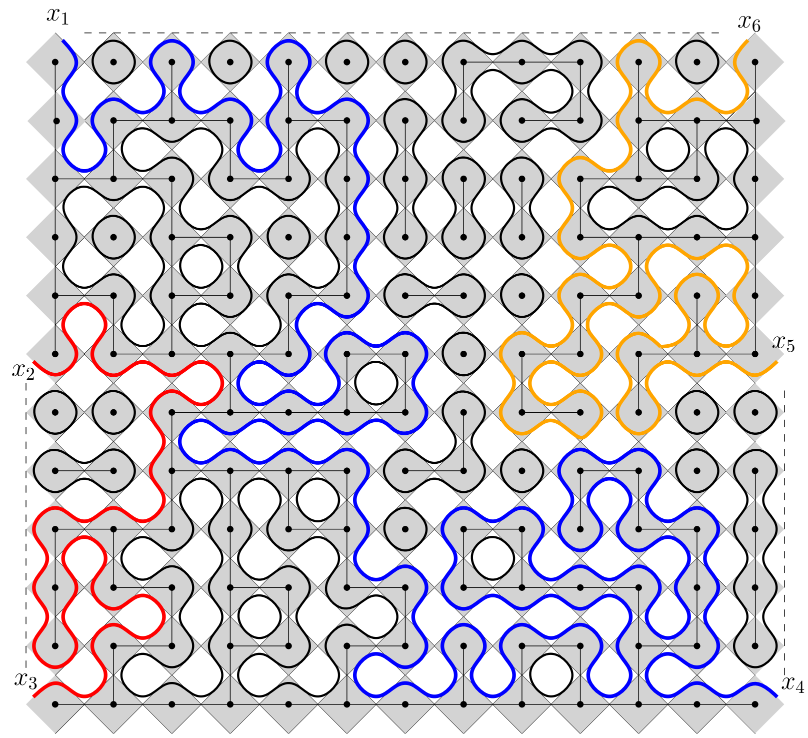

Figure 4.2: The loop representation of a configuration of the random-cluster model in a polygon with six marked points $x_1, \ldots, x_6$ on the boundary, with alternating boundary conditions. There are three interfaces connecting the marked boundary points, illustrated in red, blue, and orange, respectively.

With such boundary conditions, there are N interfaces $(\eta_1^\delta, \ldots, \eta_N^\delta)$ connecting pairwise the $2N$ boundary points $x_1^\delta, \ldots, x_{2N}^\delta$, as illustrated in Figure 4.2. These interfaces form a planar connectivity encoded in a link pattern $\mathcal{A}^\delta \in \mathrm{LP}_N$. In this section, we prove that, assuming Conjecture 4.5 is true, the collection $(\eta_1^\delta, \ldots, \eta_N^\delta)$ converges in the scaling limit to the unique global multiple SLE_κ. This would give the existence and uniqueness of global multiple SLEs for $\kappa \in (4, 6]$.

Proposition 4.7. *Suppose Conjecture 4.5 holds for some $q \in [1, 4)$. Then, for any $\alpha \in \mathrm{LP}_N$, there exists a unique global N-SLE_κ associated to α, where $\kappa \in (4, 6]$ is related to q via (4.7). In particular, for each $\alpha \in \mathrm{LP}_N$, conditionally on $\{\mathcal{A}^\delta = \alpha\}$, the collection of interfaces $(\eta_1^\delta, \ldots, \eta_N^\delta)$ in the critical random-cluster model with cluster weight q converges to the unique global multiple SLE_κ associated to α.*

It is clear that combining Proposition 4.7 with Theorem 4.6 gives Proposition 1.4.

The existence claim in Proposition 4.7 can be proven by similar arguments as Proposition 1.3 (see also [DCST17, Theorem 6]). Indeed, it follows from Proposition 4.4, inequality (4.5), and the results in [KS17] that for each $\alpha \in \mathrm{LP}_N$, conditionally on $\{\mathcal{A}^\delta = \alpha\}$, the sequence $\{(\eta_1^\delta, \ldots, \eta_N^\delta)\}_{\delta > 0}$ is relatively compact. Now, if $(\eta_1^{\delta_n}, \ldots, \eta_N^{\delta_n})$ converges weakly to $(\eta_1, \ldots, \eta_N)$ as $\delta_n \to 0$, then similar arguments as in the proof of Lemma 4.2 show that $(D_j^{\delta_n}; x_{a_j}^{\delta_n}, x_{b_j}^{\delta_n})$ converges weakly to $(D_j; x_{a_j}, x_{b_j})$ in the Carathéodory sense. Indeed, the proof of Lemma 4.2 requires two ingredients: a Russo-Seymour-Welsh (RSW) bound and an interior six-arm estimate. For the random cluster model, Proposition 4.4 gives the RSW bound, and the needed six-arm estimate follows from Conjecture 4.5 and [Wu18b]:

$$\alpha_6 = \frac{144 - (\kappa - 4)^2}{8\kappa} > 2, \quad \text{for } \kappa \in (4, 6].$$

Therefore, replacing Theorem 4.1 by Conjecture 4.5, we can use the same argument as in the proof of Lemmas 4.2 and 4.3 to conclude that the limit $(\eta_1, \ldots, \eta_N)$ is almost surely contained in the space $X_0^\alpha(\Omega; x_1, \ldots, x_{2N})$ and it is a global multiple SLE_κ associated to α.

Thus, to prove Proposition 4.7, it remains to show the uniqueness part, which we establish in the end of this section. The idea is similar to the proof of Theorem 1.2 in Section 3.3. For this purpose, we need analogues of the lemmas appearing in Sections 3.2 and 3.3.

Lemma 4.8. *Suppose Conjecture 4.5 holds for some $q \in [1, 4)$ and let $\kappa \in (4, 6]$ be the value related to q via (4.7). Let $(\Omega; x, y)$ be a bounded Dobrushin domain. Let $\Omega^L, U \subset \Omega$ be Dobrushin subdomains such that Ω^L, U, and Ω agree in a neighborhood of the arc (yx). Let $\gamma \sim \mathbb{P}(\Omega; x, y)$ and $\eta \sim \mathbb{P}(U; x, y)$. Then we have*

$$\mathbb{P}[\eta \subset \Omega^L] \geq \mathbb{P}[\gamma \subset \Omega^L].$$

In particular, Lemma 3.3 holds for the corresponding $\kappa \in (4, 6]$.

Proof. This immediately follows by combining (4.5) with Conjecture 4.5. □

Lemma 4.9. *Proposition 4.7 holds for $N = 2$.*

Proof. The existence was discussed after stating Proposition 4.7. To prove the uniqueness, we argue as in the proof of Proposition 3.2. Taking $\Omega = [0, \ell] \times [0, 1]$ and $x^L = (0, 0)$, $x^R = (\ell, 0)$, $y^R = (\ell, 1)$, $y^L = (0, 1)$, we define a Markov chain on pairs (η^L, η^R) of curves by sampling from the conditional laws: given (η_n^L, η_n^R), we pick $i \in \{L, R\}$ uniformly and resample η_{n+1}^i according to the conditional law given the other curve. However, in the current situation, we have $\kappa \in (4, 6]$, so the configuration sampled according to this rule may no longer stay in the space $X_0(\Omega; x^L, y^L, x^R, y^L)$. In this case, when resampling according to the conditional law, we sample the curves in each connected component and concatenate the pieces of curves together; see the more detailed description below Equation (4.10). However, this issue turns out to be irrelevant in the end, as we will show that, for any initial configuration $(\eta_0^L, \eta_0^R) \in X_0(\Omega; x^L, x^R, y^R, y^L)$, the corresponding Markov chain (η_n^L, η_n^R) will eventually stay in the space $X_0(\Omega; x^L, y^L, x^R, y^L)$: that is

$$\mathbb{P}\left[\exists N < \infty \text{ such that } (\eta_n^L, \eta_n^R) \in X_0(\Omega; x^L, y^L, x^R, y^L) \text{ for all } n \geq N\right] = 1. \tag{4.8}$$

Once (4.8) is proven, the uniqueness of the global 2-SLE_κ follows by repeating the proof Proposition 3.2, with Lemma 3.3 replaced by Lemma 4.8. Hence, it remains to prove (4.8).

In the Markov chain (η_n^L, η_n^R), we want to record the times when L and R are picked. Let $\tau_0^L = \tau_0^R = 0$, and for $n \geq 1$, let τ_n^R (resp. τ_n^L) be the first time after τ_{n-1}^L (resp. τ_n^R) that R (resp. L) is picked. Let

$$n_\kappa = \lceil \frac{\kappa}{8 - \kappa} \rceil + 1. \tag{4.9}$$

To prove (4.8), it suffices to show that $\eta_n^R \cap (y^L x^L) = \emptyset$ for all $n \geq \tau_{n_\kappa}^R$, because a similar property for η_n^L follows by symmetry (note also that $\tau_n^L \geq \tau_n^R$). For this purpose, we let γ^R be the SLE_κ in Ω connecting x^R and y^R. We will use the following two essential properties of γ^R:

1. By the duality property of the SLE_κ (see e.g. [Dub09] or [MS16a, Theorem 1.4]), we know that the left boundary of γ^R has the law of $\mathrm{SLE}_{\tilde{\kappa}}(\tilde{\kappa} - 4; \tilde{\kappa}/2 - 2)$ for $\tilde{\kappa} = 16/\kappa$ with two force points next to the starting point. Therefore, the left boundary of γ^R does not hit $(x^R y^R)$.

2. The curve γ^R hits $(y^L x^L)$ with positive probability, and by [AK08] and Lemma A.1, almost surely on the event $\{\gamma^R \cap (y^L x^L) \neq \emptyset\}$, the Hausdorff dimension of the intersection set satisfies

$$\dim(\gamma^R \cap (y^L x^L)) \leq 1 - \beta, \quad \text{where } \beta = \frac{8-\kappa}{\kappa}.$$

For $\tau_1^R \leq n \leq \tau_1^L - 1$, the curve η_n^R is an SLE_κ in a domain which is a subset of Ω. By Lemma 4.8, we can couple η_n^R and γ^R so that γ^R stays to the left of η_n^R almost surely. Thus, we have almost surely

$$\dim(\eta_n^R \cap (y^L x^L)) \leq \dim(\gamma^R \cap (y^L x^L)) \leq 1 - \beta.$$

In particular, for the last time before sampling the left curve, we have almost surely

$$\dim(A_1) \leq 1 - \beta \qquad \text{for} \qquad A_1 = \eta^R_{\tau_1^L - 1} \cap (y^L x^L). \tag{4.10}$$

Then, for $\tau_1^L \leq n \leq \tau_2^R - 1$, we sample η_n^L according to the conditional law given $\eta^R_{\tau_1^L - 1}$. We note that if $A_1 \neq \emptyset$, then the domain $\Omega \setminus \eta^R_{\tau_1^L - 1}$ is not connected. In this case, we sample the SLE_κ in those connected components of $\Omega \setminus \eta^R_{\tau_1^L - 1}$ which have a part of $(y^L x^L)$ on the boundary; and define η_n^L to be the concatenation of these curves. We note that, by the above observation 1, the right boundary of η_n^L only hits $(y^L x^L)$ in A_1.

Next, for $\tau_2^R \leq n \leq \tau_2^L - 1$, we sample η_n^R according to the conditional law given $\eta^L_{\tau_2^R - 1}$. Again, the curve η_n^R is an SLE_κ in a domain which is a subset of Ω and we can couple it with γ^R in such a way that γ^R stays to the left of η_n^R almost surely. Thus, we have almost surely

$$\eta_n^R \cap (y^L x^L) \subset \eta_n^R \cap A_1 \subset \gamma^R \cap A_1.$$

Combining this with (4.10), we see that almost surely

$$\dim(\eta_n^R \cap (y^L x^L)) \leq \dim(\gamma^R \cap A_1) \leq (1 - 2\beta)^+.$$

In particular, we can improve (4.10) to

$$\dim(A_2) \leq (1 - 2\beta)^+ \qquad \text{for} \qquad A_2 = \eta^R_{\tau_2^L - 1} \cap (y^L x^L), \tag{4.11}$$

almost surely. Iterating the same argument and combining with Lemma A.1, we see that almost surely,

$$\eta_n^R \cap (y^L x^L) = \emptyset \qquad \text{for all } n \geq \tau^R_{n_\kappa}. \tag{4.12}$$

This concludes the proof. ☐

By virtue of Lemma 4.8, we may also extend Lemma 3.7 to $\kappa > 4$ under Conjecture 4.5.

Corollary 4.10. *Suppose Conjecture 4.5 holds for some $q \in [1, 4)$ and let $\kappa \in (4, 6]$ be the value related to q via (4.7). Then, Lemma 3.7 holds for any global multiple SLE_κ associated to α.*

Next, we give the proof of Proposition 4.7.

Finishing the proof of Proposition 4.7. To complete the proof of the uniqueness in Proposition 4.7, we use the same arguments as in the proof of Theorem 1.2. The assertion is true for $N = 2$ by Lemma 4.9. We let $N \geq 3$ and assume that for any $\hat{\alpha} \in \mathrm{LP}_{N-1}$, the global $(N-1)$-SLE_κ associated to $\hat{\alpha}$ is unique.

As in the proof of Theorem 1.2, we suppose that $\alpha \in \mathrm{LP}_N$ with $\{1, 2\} \in \alpha$ and $\{k, k+1\} \in \alpha$ for some $k \in \{3, \ldots, 2N-1\}$, and we let $(\eta_1, \ldots, \eta_N) \in X_0^\alpha(\Omega; x_1, \ldots, x_{2N})$ be a global N-SLE_κ associated to α. We denote by η^L (resp. η^R) the curve in the collection $\{\eta_1, \ldots, \eta_N\}$ that connects x_1 and x_2 (resp. x_k and x_{k+1}). Then, by the induction hypothesis, given η^R (resp. η^L), the conditional law of the rest of the curves is the unique global $(N-1)$-SLE_κ associated to $\alpha/\{k, k+1\}$ (resp. $\alpha/\{1, 2\}$). This gives the conditional law of η^L given η^R and vice versa. One can then use the argument from the proof of Proposition 3.2, considering Markov chains sampling η^L and η^R from their conditional laws — we only need to replace Lemma 3.3 by Lemma 4.8, and Lemma 3.4 by the following Lemma 4.11 for $N - 1$. ☐

The next technical lemma is the final ingredient that we needed to complete the proof of Proposition 4.7. It can be thought of as an analogue of Lemma 3.4. To state the lemma, we fix $\alpha \in \mathrm{LP}_N$ such that $\{1,2\} \in \alpha$ and let $(\Omega; x_1, \ldots, x_{2N})$ be a bounded polygon. Also, if $(\eta_1, \ldots, \eta_N)$ is a family of random curves with the law of a global N-SLE_κ associated to α, and if η_1 is the curve connecting x_1 and x_2, then we denote by $\mathbb{Q}_\alpha^{\{1,2\}}(\Omega; x_1, \ldots, x_{2N})$ the law of η_1.

Lemma 4.11. *Suppose Conjecture 4.5 holds for some $q \in [1,4)$. Assume furthermore that there exists a unique global N-SLE_κ associated to α, where the parameter $\kappa \in (4,6]$ is related to q via (4.7). Then, let $\Omega^L \subset V \subset U, \tilde{U} \subset \Omega$ be sub-polygons such that Ω^L and Ω agree in a neighborhood of the boundary arc $(x_2\, x_1)$, and $\mathrm{dist}(\Omega^L, \Omega \setminus V) > 0$. Finally, take two random curves $\eta \sim \mathbb{Q}_\alpha^{\{1,2\}}(U; x_1, \ldots, x_{2N})$ and $\tilde{\eta} \sim \mathbb{Q}_\alpha^{\{1,2\}}(\tilde{U}; x_1, \ldots, x_{2N})$. Then, there exists a coupling $(\eta, \tilde{\eta})$ such that $\mathbb{P}[\eta = \tilde{\eta} \subset \Omega^L] \geq \theta$, where the constant $\theta = \theta(\Omega, \Omega^L, V) > 0$ is independent of U and $\tilde{U}$.*

Proof. Let $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ (resp U^δ, $\tilde{U}^\delta$, V^δ, and $(\Omega^L)^\delta$) be a sequence of discrete polygons converging to $(\Omega; x_1, \ldots, x_{2N})$ (resp U, $\tilde{U}$, V, and Ω^L) in the Carathéodory sense. Also, let $(\eta_1^\delta, \ldots, \eta_N^\delta)$ (resp. $(\tilde{\eta}_1^\delta, \ldots, \tilde{\eta}_N^\delta)$) be the collection of interfaces in the critical random-cluster model in U^δ (resp. $\tilde{U}^\delta$) with alternating boundary conditions (1.3). By the assumptions, we know that the law of η_1^δ (resp. $\tilde{\eta}_1^\delta$) conditionally on $\{\mathcal{A}^\delta = \alpha\}$ (resp. $\{\tilde{\mathcal{A}}^\delta = \alpha\}$) converges to $\mathbb{Q}_\alpha^{\{1,2\}}(U; x_1, \ldots, x_{2N})$ (resp. $\mathbb{Q}_\alpha^{\{1,2\}}(\tilde{U}; x_1, \ldots, x_{2N})$). Thus, it is sufficient to show the assertion for η_1^δ and $\tilde{\eta}_1^\delta$.

Since Ω^L agrees with Ω in neighborhoods of x_1 and x_2, there are boundary points $y_1, x_1, z_1, y_2, x_2, z_2$ in counterclockwise order along $\partial\Omega$ such that Ω^L agrees with Ω in neighborhoods of the arcs $(y_1 z_1)$ and $(y_2 z_2)$. Now, we have free boundary conditions on the arc $(x_1^\delta x_2^\delta)$ and wired boundary conditions on the arcs $(x_2^\delta x_3^\delta)$ and $(x_{2N}^\delta x_1^\delta)$. Define $\mathcal{C}^\delta$ (resp. $\mathcal{C}_*^\delta$) to be the event that there exists an open path in $(\Omega^L)^\delta$ from $(x_2^\delta z_2^\delta)$ to $(y_1^\delta x_1^\delta)$ (resp. a dual-open path in $(\Omega^L)_*^\delta$ from $(x_1^\delta z_1^\delta)$ to $(y_2^\delta x_2^\delta)$). Then, by the domain Markov property, there exists a coupling of η_1^δ and $\tilde{\eta}_1^\delta$ such that the probability of $\{\eta_1^\delta = \tilde{\eta}_1^\delta \subset (\Omega^L)^\delta\}$ is bounded from below by the minimum of $\mathbb{P}[\mathcal{C}^\delta \cap \mathcal{C}_*^\delta]$ and $\tilde{\mathbb{P}}[\mathcal{C}^\delta \cap \mathcal{C}_*^\delta]$, where $\mathbb{P}$ and $\tilde{\mathbb{P}}$ denote the probability measures of the random-cluster model in U^δ and $\tilde{U}^\delta$ with alternating boundary conditions (1.3).

Now, as a consequence of Proposition 4.4 and inequality (4.5), we have $\mathbb{P}[\mathcal{C}^\delta \cap \mathcal{C}_*^\delta] \geq \theta(\Omega, \Omega^L, V) > 0$ (and the same for $\tilde{U}$). In particular, the lower bound $\theta(\Omega, \Omega^L, V)$ is uniform over U (resp. $\tilde{U}$) and δ. By the convergence of η_1^δ and $\tilde{\eta}_1^\delta$, we obtain a coupling of η_1 and $\tilde{\eta}_1$ such that the probability of $\{\eta_1 = \tilde{\eta}_1 \subset \Omega^L\}$ is bounded from below by $\theta(\Omega, \Omega^L, V)$. This concludes the proof. $\square$

A Appendix: Intersection of two fractals

For use in Section 4, we record in this appendix some properties of random subsets of the boundary of the unit disc $\mathbb{U}$. In spite of stating the results for $\mathbb{U}$, we may as well apply the following lemma for the domain $\Omega = [0, \ell] \times [0,1]$ as we do in Section 4, by conformal invariance of the SLE_κ.

Lemma A.1. *Suppose $\mathcal{E}$ is a random subset of $\partial\mathbb{U}$ satisfying the following: there are constants $C > 0$ and $\beta \in (0,1)$ such that, for any interval I of $\partial\mathbb{U}$,*

$$\mathbb{P}[\mathcal{E} \cap I \neq \emptyset] \leq C|I|^\beta. \tag{A.1}$$

The, for any subset $A \subset \partial\mathbb{U}$,

1. *if $\dim(A) < \beta$, then*
$$A \cap \mathcal{E} = \emptyset \quad \text{almost surely} \tag{A.2}$$
2. *if $\dim(A) \geq \beta$, then*
$$\dim(A \cap \mathcal{E}) \leq \dim(A) - \beta \quad \text{almost surely.} \tag{A.3}$$

This lemma is part of [RW18, Lemma 2.3] where the authors give a more complete description of the set $A \cap \mathcal{E}$. The above cases are sufficient to our purpose in the proof of Lemma 4.9, so we include their proofs in this appendix.

Proof of item 1. Since $\beta > \dim(A)$, for any $\epsilon > 0$, there exists a cover $\cup_i I_i$ of A such that $\sum_i |I_i|^\beta \le \epsilon$. Therefore,

$$\mathbb{P}[A \cap \mathcal{E} \neq \emptyset] \le \sum_i \mathbb{P}[I_i \cap \mathcal{E} \neq \emptyset] \le C \sum_i |I_i|^\beta \le C\epsilon,$$

almost surely. Letting $\epsilon \to 0$, we see that $\mathbb{P}[A \cap \mathcal{E} \neq \emptyset] = 0$. □

Proof of item 2. For any $\gamma > \dim(A) - \beta$, there exists a cover $\cup_i I_i$ of A such that $\sum_i |I_i|^{\beta+\gamma} < \infty$. Hence,

$$\mathbb{E}\left[\sum_i |I_i|^\gamma 1_{\{I_i \cap \mathcal{E} \neq \emptyset\}}\right] = \sum_i |I_i|^\gamma \mathbb{P}[I_i \cap \mathcal{E} \neq \emptyset] \le C \sum_i |I_i|^{\beta+\gamma} < \infty,$$

almost surely. Thus,

$$\bigcup_{i:\ I_i \cap \mathcal{E} \neq \emptyset} I_i$$

is a cover of $A \cap \mathcal{E}$ and $\sum_i |I_i|^\gamma 1_{\{I_i \cap \mathcal{E} \neq \emptyset\}} < \infty$ almost surely. Therefore, we have

$$\dim(A \cap \mathcal{E}) \le \gamma, \quad a.s.$$

This holds for any $\gamma > \dim(A) - \beta$, so we have almost surely $\dim(A \cap \mathcal{E}) \le \dim(A) - \beta$. □

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