Cardy's formula on the triangular lattice, the easy way

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Abstract

We give a simple derivation of Cardy's formula for site-percolation in the triangular lattice, as proved by Smirnov; our main goal, besides simplifying the proof, is to pinpoint why the triangular lattice is so special ...

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Introduction

The problem of determining the scaling limit of critical percolation in two dimensions is long-standing, and although there have been substantial progress in the understanding of said limit *via* the study of SLE_6 processes, convergence is known only in the case of critical site-percolation on the triangular lattice, for which the essential estimate, known as Cardy's formula, was proved by Smirnov in a much celebrated paper (see [5, 6]).

In several steps of Smirnov's proof, the specific structure of the triangular lattice is used in a seemingly essential way, and it is a very natural question to try and determine where exactly it is necessary to be in this specific case, and where the construction can be unified — and thus simplified ...

The aim of this short note is to provide a mostly self-contained re-writing of Smirnov's proof, written in such a way as to emphasize the one key part of the proof where the precise geometry of the graph is used. As such, it does not contain any new result *per se* (except for the quite anecdotal Corollary 1), but we do believe that it will provide the reader with some insight into the model.

For general background on percolation, we refer the avid reader to the books of Grimmett [2] and Kesten [3]; and besides Smirnov's papers, related work on the scaling limit of percolation includes a paper of Camia and Newman [1], where they provide the full details about the convergence of percolation cluster boundaries to SLE processes, and a paper of Smirnov and Werner [7] where they derive the values of critical exponents for 2D percolation.

We will keep the same notations as in Smirnov's paper as much as possible; at various point we will be a bit terse or even a bit sloppy, since writing the proof in full would take many more pages, be tedious to read and probably not be more informative.

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1 Notations and setup

1.1 The graph

Let G_δ be the hexagonal lattice, embedded as usual to preserve its rotational symmetry of order 3, and scaled by a factor $\delta > 0$ (which we will refer to as the *mesh* of the lattice). The vertices of G_δ all have degree 3. In all the following discussion, e will always stand for an oriented edge of G_δ — which, by a slight abuse of notation, we will denote as $e \in G_\delta$.

Let G_δ^* be the dual graph of G_δ, defined as follows: The set of vertices (resp. faces) of G_δ^* is in bijection with the set of faces (resp. vertices) of G_δ, and there is an edge between two vertices of G_δ^* if and only if the two corresponding faces of G_δ share an edge. It can be embedded in the plane by choosing each of its vertices to be the center of the corresponding regular hexagon in G_δ, in which case it is exactly the usual triangular lattice, scaled by the same factor of δ. If e is an edge of G_δ, we will denote by e^* its dual edge, *i.e.* the unique edge of G_δ^* intersecting it, oriented in such a way that the frame (e, e^*) is direct.

For each vertex x of G_δ, there are three oriented edges having x as their origin. Letting $\tau = e^{2\pi i/3}$, if e is one of these edges, we will denote the other two by $\tau.e$ and $\tau^2.e$. Seeing each edge as a complex number (the difference between the positions of its endpoint and its origin), this corresponds to complex multiplication; however, we prefer seeing it as an action of the group $\{1, \tau, \tau^2\}$ on the set of oriented edges of G_δ, since this will make the constructions below easier. When needed we will also use the notation $-e$ for the edge sharing the same endpoints as e but with opposite orientation.

Notice the following, fundamental identity: For each oriented edge $e \in G_\delta$,

$$e^* + \tau(\tau.e)^* + \tau^2(\tau^2.e)^* = 0. \tag{1}$$

Here, $\tau(\tau.e)^*$ means the product of two complex numbers, namely τ and the difference between the endpoint and the origin of $(\tau.e)^*$ — we will stick to this convention, that a dot means the action of τ on edges while no dot means complex multiplication. This identity is actually trivial (using the fact that the action of τ on edges can itself be seen as complex multiplication, the sum above is equal to $(1+\tau^2+\tau^4)e^*$), but as it will appear it is essentially the only place in the construction where we use the fact that G_δ^* is the usual, equilateral triangular lattice.

It is to be noted that the above identity actually characterizes the triangular lattice; more precisely, the sum is 0 if and only if the triangle around the origin of e is equilateral. So, it seems that the triangular lattice is the only one (apart from trivial modifications of it) in which a fully combinatorial proof like follows is possible ...

1.2 The model

Let Ω be a smooth, bounded, simply connected domain in the complex plane. We will be interested in critical site-percolation on the intersection of Ω with G_δ^*. More specifically, the question we are interested in is the following. Let A, B, C and D be four points on the boundary of Ω, in that order. For every $\delta > 0$, let Ω_δ be the largest connected component (in terms of graph connectivity) of the intersection of Ω with G_δ, and let Ω_δ^* be its dual graph. Ω_δ should be seen as a discretization of Ω at scale δ. Let A_δ, B_δ, C_δ and D_δ be the vertices of Ω_δ that are closest to A, B, C and D respectively.

The model we are considering is now critical site-percolation on Ω_δ^*, *i.e.*, each vertex of Ω_δ^* is taken to be open with probability $p_c = 1/2$, independently of all the others. Let $C_\delta(\Omega, A, B, C, D)$ be the event that there is an open crossing in Ω_δ^*, between the intervals $A_\delta B_\delta$ and $C_\delta D_\delta$ of its boundary. Standard Russo-Seymour-Welsh estimates (cf. [2] for instance) tells us that at criticality, the probability of $C_\delta(\Omega, A, B, C, D)$ is bounded away from both 0 and 1 as δ goes to 0; The main result we prove here is the following:

Theorem 1 (Smirnov) *The probability of the event $C_\delta(\Omega, A, B, C, D)$ has a limit $f(\Omega, A, B, C, D)$ as δ goes to 0. Moreover, the limit is conformally invariant, in the following sense: If Φ is a conformal map from Ω to another simply connected domain $\Omega' = \Phi(\Omega)$, and extends continuously to $\partial\Omega$, then*

$$f(\Omega, A, B, C, D) = f(\Phi(\Omega), \Phi(A), \Phi(B), \Phi(C), \Phi(D)).$$

As will appear naturally in the proof and was first pointed to by Carleson, f has a most simple expression in the case where Ω itself is an equilateral triangle with vertices A, B and C and D on the interval (CA): Then $f(\Omega, A, B, C, D) = |CD|/|CA|$. By conformal invariance, this gives the value of f for every conformal rectangle.

2 The proof

2.1 General framework

The framework of the proof here is the same as in Smirnov's proof, but we describe it in some detail for sake of self-containedness. Recall that Ω_δ is a discrete approximation of Ω; for every point z be a point of the Ω_δ lying inside one of the triangles of Ω_δ^*, we will denote by $E_{A,\delta}(z)$ the event that there exists a simple path of open vertices in Ω_δ^*, separating A_δ and z from B_δ and C_δ — and $E_{B,\delta}(z)$, $E_{C,\delta}(z)$ similarly, with obvious circular permutations of the letters. Let $H_{A,\delta}(z)$ (resp. $H_{B,\delta}(z)$, $H_{C,\delta}(z)$) be the probability of $E_{A,\delta}(z)$ (resp. $E_{B,\delta}(z)$, $E_{C,\delta}(z)$). From now on, in many cases where this does not introduce confusion, we will omit the mention of δ in those notations.

From Russo-Seymour-Welsh theory, we obtain the following: There are two positive constants K_E and ε_E such that, for every $\delta > 0$ and any two points z and z' in Ω_δ (up to the boundary of the domain),

$$\left|H_{A,\delta}(z') - H_{A,\delta}(z)\right| \leqslant K_E \left|z' - z\right|^{\varepsilon_E} \tag{2}$$

and a similar bound for H_B and H_C. Hence, if we suitably extend these functions continuously to Ω, we obtain a family of uniformly Hölder maps from Ω to $\mathbb{R}$. The family is then relatively compact with respect to uniform convergence, and it is hence possible to extract a subsequence $(H_{A,\delta_n}, H_{B,\delta_n}, H_{C,\delta_n})_{n>0}$, with $\delta_n \to 0$, which converges uniformly to a triple of Hölder maps (h_A, h_B, h_C) from $\overline{\Omega}$ to $[0,1]$, with the same exponent and norm as the H_A's.

If we can identify the triple (h_A, h_B, h_C) uniquely, we will obtain convergence of the family $(H_{A,\delta}, H_{B,\delta}, H_{C,\delta})$ as δ goes to 0. Then, making the additional remark that $E_{C,\delta}(D_\delta)$ is the same event as $C_\delta(\Omega, A, B, C, D)$, this will conclude the proof of the first statement in Theorem 1.

The way we will do that is slightly different from Smirnov's, the difference being essentially in the determination of the boundary conditions (see Subsection 2.4). Introduce the following two functions:

$$H_\delta(z) := H_{A,\delta}(z) + \tau H_{B,\delta}(z) + \tau^2 H_{C,\delta}(z) \qquad S_\delta(z) = H_{A,\delta}(z) + H_{B,\delta}(z) + H_{C,\delta}(z).$$

(This is slightly different from Smirnov's H, but the idea is exactly the same and the version we use here is more symmetric.) They also form a family of uniformly Hölder maps, and the subsequence (H_{δ_n}) (resp. S_{δ_n}), with the same (δ_n) as were introduced previously, converges to

$$h := h_A + \tau h_B + \tau^2 h_C, \qquad \text{resp.} \quad s := h_A + h_B + h_C.$$

The key step of the proof, to which the next two sections are devoted, is to prove that the functions h and s are holomorphic on Ω. Subsection 2.4 then concludes the proof. Notice though that since s is both holomorphic and real-valued, it has to be constant, and it is easy to see from boundary conditions that it is actually equal to 1:

Corollary 1 *As the mesh of the discretization vanishes, the sum of the probabilities of $E_{A,\delta}(z)$, $E_{B,\delta}(z)$ and $E_{C,\delta}(z)$ converges to 1, uniformly in $z \in \overline{\Omega}$.*

The exact same proof works for any triangulation in which Russo-Seymour-Welsh holds, so this fact seems to be a fundamental property of critical two-dimensional percolation (and *might* be the key to understanding universality in this particular, limited case, though this is hardly even speculative). As of this time, no direct, combinatorial proof of the corollary seems to be known.

2.2 Discrete integration

The theorem best adapted in the present case to prove that h is indeed holomorphic is Morera's theorem, which we recall without proof.

Theorem 2 (Morera) *Let Ω be a simply connected domain of the complex plane, and let $f : \Omega \to \mathbb{C}$ be a continuous function. Then f is holomorphic if, and only if, for every simple, closed, smooth curve γ contained in Ω, the integral of f along γ vanishes.*

So, let γ be a simple, closed, smooth curve contained in Ω, and for every $\delta > 0$ let γ_δ be a discretization of γ contained in Ω_δ, *i.e.* a finite chain $(\gamma_\delta(k))_{0 \leqslant k \leqslant N_\delta}$ of pairwise distinct vertices of G_δ, ordered in the positive direction, such that for every index k, $\gamma_\delta(k)$ and $\gamma_\delta(k+1)$ are nearest neighbors, and chosen in such a way that the Hausdorff distance between γ_δ and δ goes to 0 with δ. Notice that N_δ can be taken of order δ^{-1}, which we shall assume from now on.

Since the subsequence (δ_n) is chosen so that h_{δ_n} converges uniformly to h, we have the following fact: As n goes to infinity,

$$I_n(\gamma) := \sum_{k=0}^{N_{\delta_n}} \frac{H_{\delta_n}(\gamma_{\delta_n}(k)) + H_{\delta_n}(\gamma_{\delta_n}(k+1))}{2} (\gamma_{\delta_n}(k+1) - \gamma_{\delta_n}(k)) \to \oint_\gamma h(z) \mathrm{d}z. \tag{3}$$

So, to prove that h is holomorphic, it is sufficient to show that $I_n(\gamma)$ goes to 0 as n goes to infinity.

The discrete curve γ_δ surrounds a finite family of faces of G_δ, which we shall denote by $\mathring{\gamma}_\delta$. Besides, for every (oriented) edge $e = xy$ in G_δ, define the following notations:

$$H_\delta(e) := \frac{H_\delta(x) + H_\delta(y)}{2}, \qquad \partial_e H := H(y) - H(x)$$

(without dividing by the length of e). Last, if f is a face of G_δ, let ∂f be its oriented boundary, seen as a set of oriented edges. With these notations, we get a shorter notation for I_n (where $e \in \gamma_\delta$ means that e is of the form $\gamma_{\delta_n}(k)\gamma_{\delta_n}(k+1)$ for some k) and the following identity:

$$I_n(\gamma) = \sum_{e \in \gamma_\delta} e H_\delta(e) = \sum_{f \in \mathring{\gamma}_\delta} \sum_{e \in \partial f} e H_\delta(e). \tag{4}$$

Indeed, in the last equality, each boundary term is obtained exactly once with the correct sign, and each interior term appears twice with opposite signs — for every edge e, $H_\delta(-e) = H_\delta(e)$.

If f is a face of G_δ, let $c(f)$ be its center (it is a vertex of G_δ^*). The sum of $eH(e)$ around f can be rewritten in the following fashion: If $x_0, x_1, \ldots, x_{|f|} = x_0$ are the vertices of f, we can "integrate by parts" (which is the same as re-indexing part of the sum — we do it explicitly this time, and shall do it quietly a few more times below) and obtain

$$\begin{aligned}
\sum_{e\in\partial f} eH_\delta(e) &= \sum_{k=0}^{|f|-1} (x_{k+1}-x_k)\frac{H(x_{k+1})+H(x_k)}{2} \\
&= \sum_{k=0}^{|f|-1} (x_{k+1}-c(f))\frac{H(x_{k+1})+H(x_k)}{2} - \sum_{k=0}^{|f|-1} (x_k-c(f))\frac{H(x_{k+1})+H(x_k)}{2} \\
&= \sum_{k=0}^{|f|-1} (x_k-c(f))\frac{[H(x_{k+1})-H(x_k)]+[H(x_k)-H(x_{k-1})]}{2} \\
&= \sum_{k=0}^{|f|-1} \left(\frac{x_k+x_{k+1}}{2}-c(f)\right)[H(x_{k+1})-H(x_k)].
\end{aligned}$$

Putting this back in the previous sum, notice that for x, y nearest neighbors in the interior of γ, the term $H(y)-H(x)$ appears twice, once for each face of G_δ adjacent to xy, and that the factors $(x+y)/2$ cancel between them, leaving only the difference between the centers of the faces, *i.e.* the dual edge of xy.

On the boundary, we obtain of the order of δ^{-1} terms, each being of order $\delta^{1+\varepsilon_H}$, so the contribution of the boundary terms goes to 0 with δ, and we finally obtain

$$I_n(\gamma) = \frac{1}{2}\sum_{e\subset\gamma_\delta} e^*\partial_e H + o(1), \tag{5}$$

with the slight abuse of notation that $e \subset \gamma_\delta$ means that e is an edge of $H\delta$ lying in the interior of γ. We will keep this notation from now on.

For every edge $e = xy$ of Ω_δ let $P_{A,\delta}(e)$ (and similarly P_B and P_C) be the probability that $E_{A,\delta}(y)$ is satisfied, but $E_{A,\delta}(x)$ is not. Again by Russo-Seymour-Welsh, we get the following estimate, uniformly in δ and e:

$$P_{A_\delta}(e) \leqslant K_P\delta^{\varepsilon_P}. \tag{6}$$

Besides, we have for every edge e:

$$\partial_e H_A = P_A(e) - P_A(-e).$$

So, replacing H by its definition in (5), and re-indexing the sum to obtain each oriented edge in exactly one term, we get the following expression for the discrete contour integral:

$$I_n(\gamma) = \sum_{e\subset\gamma_\delta} e^*\left[P_A(e)+\tau P_B(e)+\tau^2 P_C(e)\right] + o(1). \tag{7}$$

2.3 Color-swapping

The main combinatorial tool that we will use in what follows, and in fact the only place where the fact that the model is critical percolation at $p_c = 1/2$, is the following (stated here with our notations):

Proposition 1 (Smirnov) *For every edge e of Ω_δ, we have the following identities:*

$$P_A(e) = P_B(\tau.e) = P_C(\tau^2.e).$$

We refer the reader to [6] for the (elementary, but very clever) proof of this result; said proof does not actually use the fact that we are considering the triangular lattice, or even the fact that 1/2 is the critical parameter: It extends to site-percolation with parameter 1/2 on any planar triangulation. Using this identity in (7), we obtain

$$\begin{aligned} I_n(\gamma) &= \sum_{e\subset\gamma_\delta} e^* \left[P_A(e) + \tau P_A(\tau^2.e) + \tau^2 P_A(\tau.e)\right] + o(1) \\ &= \sum_{e\subset\gamma_\delta} (e^* + \tau(\tau.e)^* + \tau^2(\tau^2.e)^*) P_A(e) + o(1), \end{aligned}$$

where the last step is obtained by re-indexing the sum one last time. Each term in the sum is then equal to 0, by Equation (1), so we get $I_n(\gamma) = o(1)$ which is exactly what we needed. It follows that the limit h is holomorphic on Ω.

Replacing H with S in the above computation, it is easy to check that one gets the sum, over the same family of oriented edges, of $(e^* + (\tau.e)^* + (\tau^2.e)^*)P_A(e)$; but the identity

$$e^* + (\tau.e)^* + (\tau^2.e)^* = 0 \tag{8}$$

is true in any lattice, as it is the sum of the oriented edges of the triangle around the origin of e. Notice how this does not use the exact details of the triangulation anymore, only Russo-Seymour-Welsh estimates, so it holds on any triangulation for which those hold, as announced.

2.4 Boundary conditions

The last step we have to perform, now that we know that h is holomorphic, is to identify enough boundary conditions to specify it uniquely. This can be done in a very simple way, as follows.

Let z be a point on the boundary of Ω_δ lying between B and C. It is clear that $h_A(z) = 0$, and it is true that $h_B(z) + h_C(z) = 1$ (either from Corollary 1, or by looking at the discrete exploration process started at A and aiming at the boundary interval $[BC]$, for which $H_B(z)$ (resp. $H_C(z)$) is the probability that it touches $[zC]$ before (resp. after) $[Bz]$). Hence, $h(z)$ lies on the interval $[\tau, \tau^2]$ of the complex plane. Besides, $h(B) = \tau$ and $h(C) = \tau^2$, so h induces a continuous map from the boundary interval $[BC]$ of Ω onto $[\tau, \tau]$. By Russo-Seymour-Welsh yet again, h is one-to-one on this boundary interval.

Similarly, h induces a bijection between the boundary interval $[AB]$ (resp. $[CA]$) of Ω and the complex interval $[1, \tau]$ (resp. $[\tau^2, 1]$), so putting the pieces together we see that h is a holomorphic map from Ω to $\mathbb{C}$ which extends continuously to $\bar{\Omega}$ and induces a continuous bijection between $\partial\Omega$ and the boundary of the triangle with vertices at 1, τ and τ^2.

From standard results of complex analysis ("principle of corresponding boundaries", cf. for instance Theorem 4.3 in [4]), this implies that h is actually a conformal map from Ω to the interior of the same triangle. But we know that h maps A (resp. B, C) to 1 (resp. τ, τ^2), and this determines it uniquely. In other words, there is only one possible limit for the triple

(H_A, H_B, H_C) as δ goes to 0, which gives conformal invariance for free and concludes the proof of Theorem 1.

As a corollary of Theorem 1, we get a nice expression for h_A: If $\Phi_{\Omega,A,B,C}$ is the conformal map from Ω to the triangle mapping A, B and C as previously (which means of course that $\Phi_{\Omega,A,B,C} = h$) then

$$H_{A,\delta}(z) \to \frac{2\Re(\Phi_{\Omega,A,B,C}(z)) + 1}{3}.$$

If in particular Ω is the equilateral triangle itself, then h is the identity map and we obtain Cardy's formula in Carleson's form: if $D \in [CA]$ then

$$P_{p_c}\left([AB] \leftrightarrow [CD]\right) = \frac{|CD|}{|AB|}.$$

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