M2R maths Grenoble
2016–2017
Tutorials

Continuous-time Markov chains

Exercise 1.1 From discrete to continuous time Markov chains.
Suppose that $P = (p(x,y))_{x,y\in S}$ is the transition matrix of a discrete time Markov chain with countable state-space S. The most natural way to make it into a continuous time Markov chain is to have it take steps at the event times of a Poisson process of intensity 1 independent of the discrete chain. In other words, the times between transitions are independent random variables with a unit exponential distribution. The forgetfulness property of the exponential distribution is needed in order that the process have the Markov property.

1. Show that the transition function of the continuous time chain described above is given by:

$$p_t(x,y) = e^{-t}\sum_{k=0}^{+\infty}\frac{t^k}{k!}p_k(x,y) \tag{1}$$

where $p_k(x,y)$ are the k-step transition probabilities for the discrete time chain.

2. Show that (1) satisfies the Chapman-Kolmogorov equations, i.e for all $s,t \geq 0$ and $x,y \in S$:

$$p_{s+t}(x,y) = \sum_{z\in S} p_s(x,z)p_t(z,y)\ .$$

3. Show that the Q-matrix is given by

$$Q = P - I\ .$$

Exercise 1.2 The two-states Markov chain
Suppose that $S = \{0,1\}$. Consider a Q-matrix that is given by

$$\begin{pmatrix} -\beta & \beta \\ \delta & -\delta \end{pmatrix}$$

where $\beta,\delta > 0$. Show that the corresponding transition matrix is

$$P_t = \begin{pmatrix} \frac{\delta}{\beta+\delta} + \frac{\beta}{\beta+\delta}e^{-t(\beta+\delta)} & \frac{\beta}{\beta+\delta}(1-e^{-t(\beta+\delta)}) \\ \frac{\delta}{\beta+\delta}(1-e^{-t(\beta+\delta)}) & \frac{\beta}{\beta+\delta} + \frac{\delta}{\beta+\delta}e^{-t(\beta+\delta)} \end{pmatrix}$$

Exercise 1.3 Pure birth chain and explosion
Let $S = \mathbb{N}$ and

$$q(i,j) = \begin{cases} -\beta_i \text{ si } j = i \\ \beta_i \text{ si } j = i+1 \\ 0 \text{ otherwise,} \end{cases}$$

where $\beta_i > 0$ for each i. Let $(\tau_i)_{i\geq 0}$ be independent exponentially distributed random variables with τ_i having parameter β_i. Define:

$$T := \sum_{i=0}^{\infty} \tau_i \in [0, +\infty]$$

1. Show that if $\sum_{i=0}^{\infty} \frac{1}{\beta_i} < \infty$, $T < \infty$ a.s and if $\sum_{i=0}^{\infty} \frac{1}{\beta_i} = \infty$, $T = \infty$ a.s.
2. What does this imply for the existence of a Markov chain with Q-matrix Q as above ?

Exercise 1.4 MCMC I: Metropolis

Let Ψ be an irreducible Q-matrix on a finite state-space S and π be a probability distribution on S. We want to modify the markov chain given by Ψ to obtain a new markov chain whose stationary distribution is π. We shall adopt the following method: when at state x, a candidate move is generated with rates $\Psi(x, \cdot)$. If the proposed state is y, the move is accepted with probability $a(x, y)$ for some weight $a(x, y)$, and rejected (i.e the chain stays at x) with probability $1 - a(x, y)$.

1. Show that the Q-matrix of the new chain is given by:

$$Q(x,y) = \begin{cases} \Psi(x,y)a(x,y) & \text{if } y \neq x \\ -\sum_{z:z\neq x} \Psi(x,z)a(x,z) & \text{if } y = x \end{cases}$$

2. We want to find weights $a(x, y)$ such that Q is reversible with respect to π and we reject the candidate moves as seldom as possible. Show that this leads to the choice:

$$a(x,y) = \frac{\pi(y)\Psi(y,x)}{\pi(x)\Psi(x,y)} \wedge 1$$

   This is called the (continuous-time) *Metropolis chain* associated to π and Ψ.

3. Suppose that you know neither the vertex set nor the edge set of a finite connected graph, but you are able to perform a random walk on it and you want to approximate the uniform distribution on the vertex set by running a Metropolis chain for a long time. Make explicit the Q-matrix of the chain.

Exercise 1.5 MCMC II: Glauber

Let S_0 and V be finite sets and suppose that S is a non-empty subset of S_0^V. We shall call elements of V the *vertices* and elements of S the *configurations*. Let π be a probability distribution on S. The *Glauber dynamics for* π is a reversible Markov chain on S defined as follows. Vertices are equipped with independent Poisson processes with rate 1, let us name them "clocks". When at configuration ω, as soon as a clock rings at vertex $v \in V$, a new state for the configuration at v is chosen according to the measure π conditioned on the configurations which agree with ω at all vertices different from v.

1. For a configuration $\eta \in S$ and a vertex $v \in V$, let $S(\eta, v)$ denote the configurations agreeing with η everywhere except perhaps at v. Show that the Q-matrix of this chain is given by:

$$Q(\eta, \eta') = \begin{cases} \frac{\pi(\eta')}{\pi(S(\eta,v))} & \text{if } \eta' \in S(\eta, v) \text{ and } \eta' \neq \eta \\ 0 & \text{if } \eta \text{ and } \eta' \text{ differ at least on two coordinates} \\ -|V| + \pi(\eta) \sum_{v \in V} \frac{1}{\pi(S(\eta,v))} & \text{if } \eta = \eta' \end{cases}$$

2. Check that Q is reversible with respect to π. Is it irreducible ?

3. The *Ising model* on the graph $G = (V, E)$ is the family of probability distributions on $S = \{-1, 1\}^V$ defined by:

$$\pi_\beta(\eta) = \frac{1}{Z(\beta)} e^{-\beta H(\eta)} ,$$

where $\beta \geq 0$ is a parameter, H is the hamiltonian defined by

$$H(\eta) = -\frac{1}{2} \sum_{\substack{u,v \in V \\ \{u,v\} \in E}} \eta(u)\eta(v)$$

and $Z(\beta)$ is the normalization constant, called the partition function:

$$Z(\beta) = \sum_{\eta \in S} e^{-\beta H(\eta)} .$$

Make explicit the Q-matrix of the Glauber dynamics for π_β. Is it irreducible ?

Exercise 1.6 Coupling and mixing time

Let Q be an irreducible Q-matrix on a finite state-space S and denote by π its invariant probability measure. Recall the notion of total variation distance between two probability measures on S:

$$d_{VT}(\mu, \nu) = \sup_{A \subset S} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{x \in S} |\mu(x) - \nu(x)|$$

1. Suppose that (X, Y) is a random vector with values in S^2 such that X (resp. Y) has distribution μ (resp. ν). Show that:

$$d_{VT}(\mu, \nu) \leq \mathbb{P}(X \neq Y) .$$

2. One way to measure the convergence to equilibrium is in the sense of total variation. Define:

$$d(t) := \max_{x \in S} d_{VT}(p_t(x, \cdot), \pi)$$

and:

$$\bar{d}(t) := \max_{x,y \in S} d_{VT}(p_t(x, \cdot), p_t(y, \cdot)) .$$

Show that d is non-increasing, that

$$d(t) \leq \bar{d}(t) \leq 2d(t) ,$$

and

$$d(t) = \max_{\mu} \; d_{VT}(\mathbb{P}_\mu(X_t = \cdot), \pi) \; .$$

For $\varepsilon < \frac{1}{2}$, one usually defines the *mixing time* $t_{mix}(\varepsilon)$ as follows:

$$t_{mix}(\varepsilon) := \inf\{t : d(t) \leq \varepsilon\}$$

3. Suppose that $(X_t, Y_t)_{t\geq 0}$ is a stochastic process with values in S^2 such that (X_t) and (Y_t) are random walks with Q-matrix Q (this is called a *coupling of two markov chains* with Q-matrix Q). Suppose also that X and Y stay together as soon as they meet:

$$\text{if } X_s = Y_s \text{ then } X_t = Y_t \text{ for any } t \geq s \; .$$

and define the *coupling time* τ of X and Y as:

$$\tau := \inf\{t \geq 0 : X_t = Y_t\} \; .$$

Let μ (resp. ν) be the distribution of X_0 (resp. Y_0). Show that:

$$d_{VT}(\mu e^{tQ}, \nu e^{tQ}) \leq \mathbb{P}(\tau > t)$$

4. One may define a coupling as follows, X starting from x and Y starting from y: we let the two chains evolve independently (each one according to Q) until they meet, after which we let them evolve according to Q, but staying equal. Write down the Q-matrix corresponding to this Markov chain on S^2.

5. Now, we suppose that Q defines a simple random walk on the circle $\mathbb{Z}/n\mathbb{Z}$ (with jump rates $1/2$ to the right and to the left). Using the coupling of question 4, and Markov's inequality on the coupling time, show that:

$$t_{mix}(\varepsilon) \leq \frac{n^2}{8\varepsilon} \; .$$

Exercise 1.7 Spectral representation of a finite state-space reversible Markov chain
Let π be a positive probability measure on $S = \{1, \ldots, n\}$ and let Q be an irreducible $n \times n$ Q-matrix reversible with respect to π. Denote by D the diagonal matrix whose diagonal elements are $\sqrt{\pi(i)}$.

1. Show that the matrix $S = DQD^{-1}$ is symmetric. Thus, there exists an orthonormal matrix U and a diagonal matrix Λ such that:

$$S = U \Lambda U^T$$

and the diagonal elements of Λ can be chosen to be non-increasing: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

2. Show that $\lambda_2 < \lambda_1 = 0$.

3. Show that the column vectors of the matrix DU constitute a basis of eigenvectors for Q^T, with the k-th column associated to the eigenvalue λ_k.

4. Show that the column vectors of the matrix $D^{-1}U$ constitute a basis of eigenvectors for Q, with the k-th column associated to the eigenvalue λ_k.

5. Let X_t be a continuous-time Markov chain with Q-matrix Q. Show the *spectral representation formula*: for any i, j and any $t \geq 0$,
$$p_t(i,j) := \mathbb{P}_i(X_t = j) = \pi(i)^{-1/2}\pi(j)^{1/2}\sum_{k=1}^{n} e^{-\lambda_k t} u_{ik} u_{jk} \ .$$

6. When n is fixed and t goes to infinity, describe the rate of convergence of $p_t(i,j)$ towards $\pi(j)$.

7. Make explicit the rate of convergence of $\|p_t(i,.) - \pi\|_2$ towards zero when Q defines a simple random walk on the circle $\mathbb{Z}/n\mathbb{Z}$. Give also lower and upper bounds on the mixing time (defined in Exercise 1.6).

Exercise 1.8 M/M/1 queue
Consider the following model for a queue. Customers arrive according to a Poisson process with rate $\lambda > 0$ and form a queue: there is a single server (i.e only one customer can be served at a given time), and the service times are independent, exponential random variables with rate $\mu > 0$. We denote by X_t the number of customers in the queue at time t, the number X_0 being independent from the arrivals and service times at positive times.

1. Compute the Q-matrix of this chain and determine the invariant measures.

2. What happens, in the asymptotic sense, when $\lambda \geq \mu$? And when $\lambda < \mu$?

3. Suppose that $\lambda < \mu$. Show that the asymptotic proportion of the time that the server is busy equals almost surely:
$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{1}_{X_t>0}\, dt = \frac{\lambda}{\mu} \ .$$

Exercise 1.9 The graphical representation without graphics
Let S be a topological space equipped with its Borel sigma-field and Ω be the set of càdlàg functions from $\mathbb{R}^+$ to S equipped with the σ-field generated by the maps $\omega \mapsto \omega(t)$ for $t \geq 0$. Let $(E, \mathcal{E}, \mu)$ be a measured space with μ finite on E_n and $\bigcup_n E_n = E$. Consider $\tilde{N}$ a poisson random measure on $\mathbb{R} \times E$ with intensity $\lambda \otimes \mu$, λ being the Lebesgue measure on $\mathbb{R}$. Let $\tilde{\Omega}$ denote the set of counting measures on $\mathbb{R} \times E$ which are finite on each set of the form $]s,t] \times E_n$. Define, for any $\tilde{\omega}$ in $\tilde{\Omega}$:
$$\theta_s\tilde{\omega}(A) = \tilde{\omega}(\{(t,x) \in \mathbb{R} \times E \,:\, (t-s, x) \in A\}) \ .$$

For any $t \geq 0$, let ϕ_t be a measurable function from $\Omega \times S$ to S. Suppose that:

(i) for every $\tilde{\omega} \in \tilde{\Omega}$ and $x \in S$, $s \mapsto \phi_s(\tilde{\omega}, x)$ is càdlàg.

(ii) $\forall t \geq 0, x \in S$ $\phi_t(0, x) = x$

(iii) $\forall (\tilde{\omega}, x) \in \tilde{\Omega} \times S$, $\forall t \geq 0$ $\phi_t(\tilde{\omega}, x) = \phi_t(\tilde{\omega}|_{]0,t] \times E}, x)$

(iv) $\forall s, t \geq 0$ $\phi_{t+s}(\tilde{\omega}, x) = \phi_t(\theta_s \tilde{\omega}, \phi_s(\tilde{\omega}, x))$

1. For x in S, let $X_t^x := \phi_t(\tilde{N}, x))$, $X^x := (X_t^x)_{t \geq 0}$ and $\tilde{\mathcal{F}}_t = \sigma(\tilde{N}|_{]0,t] \times E})$. Show that for any bounded measurable function f on Ω,

$$\mathbb{E}[f(X^x) | \tilde{\mathcal{F}}_t] = \mathbb{E}[f(X^y)]|_{y = X_t^x}$$

2. Let $\mathbb{P}_x$ denote the distribution (on Ω) of X^x. Show that $(\mathbb{P}_x)_{x \in S}$ defines a continuous time Markov chain with respect to the canonical filtration $(\mathcal{F}_t)$ of Ω.

3. Suppose now that S is countable and μ is a finite measure. Let

$$T := \inf\{t > 0 : \tilde{N}|_{]0,t] \times E} \neq 0\} ,$$

   Show that T has exponential distribution with parameter $\mu(E)$ and that there is a random variable Y with values in E, independent from T and with distribution $\mu(.)/\mu(E)$ such that almost surely, $\tilde{N}|_{]0,T] \times E} = \delta_{(T,Y)}$.

4. Show that when t goes to zero and when $y \neq x$,

$$\mathbb{P}(X_t^x = y) = t\mu(E) \mathbb{P}(\phi_1(\delta_{(1,Y)}, x) = y) + O(t^2) .$$

Exercise 1.10 Irreducibility in continuous time

Let Q be the Q-matrix of a continuous-time Markov chain on a countable state-space S such that $c(x) := -Q(x, x) < \infty$ for any x.

1. Using the Chapman-Kolmogorov equations, show that if $q(x, y) > 0$, then $p_t(x, y) > 0$ for any $t > 0$.

2. Show that if there are $n \in \mathbb{N}$ and $x_0 = x, x_1 \ldots, x_n = y$ in S such that for any i $q(x_i, x_{i+1}) > 0$, then $p_t(x, y) > 0$ for any $t > 0$.

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Brownian motion

In all the exercises below, B is a standard Brownian motion, $\Omega = C[0, +\infty)$ is the set of continuous functions on $\mathbb{R}^+$, $\mathcal{F}$ is the sigma-algebra generated by the coordinate maps and P^x is the distribution of $x + B(.)$ on $(\Omega, \mathcal{F})$. Let also $p_t(x, .)$ be the density of $\mathcal{N}(x, t)$ and $p_t(x) = p_t(0, x)$.

Exercise 2.1 Markov property
Consider the following hitting times:

$$\tau_1 = \inf\{t > 0 : \ B(t) = 0\},$$

$$\tau_2 = \inf\{t > 1 : \ B(t) = 0\},$$

$$\tau_3 = \sup\{t < 1 : \ B(t) = 0\} .$$

Check the following relations:

$$P^x(\tau_2 \le t) = \int_{\mathbb{R}} p_1(x, y) P^y(\tau_1 \le t - 1)\, dy , \quad t \ge 1$$

$$P^0(\tau_3 \le t) = \int_{\mathbb{R}} p_t(0, y) P^y(\tau_1 > 1 - t)\, dy , \quad 0 \le t < 1$$

Exercise 2.2 Brownian Bridge
Define $X(t) = B(t) - tB(1)$ for $t \in [0, 1]$.

1. Show that X is a Gaussian process, compute its covariance and show that X is independent from $B(1)$.
2. Show that the density of $(X(t_1), \dots, X(t_n))$ for $0 < t_1 < \dots < t_n < 1$ is
$$g(x_1, \dots, x_n) = p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) \dots p_{t_n - t_{n-1}}(x_n - x_{n-1}) p_{1-t_n}(-x_n) .$$
3. Show that $(X_{1-t})_{t\in[0,1]}$ has the same distribution than X.
4. Prove that the distribution of $(B(t_1), \dots, B(t_n))$ conditionally on $\{|B(1)| \le \varepsilon\}$ converges to the distribution of $(X(t_1), \dots, X(t_n))$ when ε goes to zero.

Exercise 2.3 Blumenthal's $0 - 1$ law
Let $(\mathcal{F}_t)$ denote the canonical right-continuous filtration on $(\Omega, \mathcal{F})$: $\mathcal{F}_t^0$ is the sigma-algebra generated by the coordinate maps $\omega \mapsto \omega(s)$ for $s \le t$ and $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t^0$.

1. Show that if Y is a bounded random variable, for every x,
$$E^x(Y|\mathcal{F}_s) = E^x(Y|\mathcal{F}_s^0) \quad P^x - a.s.$$

2. Show that if $A \in \mathcal{F}_0$, then $P^0(A) \in \{0, 1\}$.

3. Use this result to show that P^0-a.s, $0 = \inf\{t \geq 0 \text{ s.t. } \omega(t) > 0\} = \inf\{t \geq 0 \text{ s.t. } \omega(t) = 0\}$.

Exercise 2.4 Strong Markov property
If τ is a finite stopping time, show that $(B(\tau + t) - B(\tau))_{t \geq 0}$ is a brownian motion independent from $\mathcal{F}_\tau$.

Exercise 2.5 Reflection principle
Let $M_t = \max_{0 \leq s \leq t} B_s$.

1. Show that if $a > 0$ and $b < a$, then for any $t > 0$,

$$\mathbb{P}(M_t > a,\ B_t < b) = \mathbb{P}(B_t > 2a - b)$$

   and compute the joint density of (M_t, B_t).

2. Show that $M_t - B_t$ has the same distribution as $|B_t|$.

3. Show that M_t has the same distribution as $|B_t|$.

Exercise 2.6 Reflected Brownian motion

1. Let f be a positive measurable function on $\mathbb{R}^n$. Let s and $t_1, \ldots, t_n$ be fixed positive real numbers. Let $g(B_s) = \mathbb{E}[f(|B_{s+t_1}|, \ldots, |B_{s+t_n}|)|B_s]$. Using the fact that B and $-B$ have the same distribution, show that g is $\sigma(|B_s|)$-measurable.

2. Show that $|B_t|$ is a Markov process and write down its transition kernel.

Exercise 2.7 Monotony and local maxima

1. Let f be a continuous function on an interval $[a; b]$ which is monotone in no subinterval of $[a; b]$. Show that it has a local maxima in $(a; b)$.

2. Let B be a standard Brownian motion. Show that with probability one,

   (a) B is monotone in no interval,
   (b) The set of times at which local maxima occur is dense,
   (c) Every local maximum is strict,
   (d) The set of times at which local maxima occur is countable.

Exercise 2.8 The zero set of Brownian motion I
For each $a \geq 0$, define

$$\tau_a(\omega) = \inf\{t \geq a : \omega(t) = 0\} .$$

and

$$Z(\omega) = \{t \in \mathbb{R}^+ : Z(t) = 0\}$$

be the set of zeroes of ω.

1. Show that every point of $Z(\omega) \setminus \{\tau_a(\omega) : a \in \mathbb{Q}^+\}$ is a limit from the left of points of $Z(\omega)$.
2. Let $Y = \mathbf{1}_A$ with

$$A = \{\omega : \omega(t_n) = 0 \text{ for some sequence } t_n \text{ with } t_n \downarrow 0\}$$

   Show that P^x-a.s

$$E^x(Y \circ \theta_{\tau_a} | \mathcal{F}_{\tau_a}) = 1 .$$

3. Show that for each a, P^x-almost surely, τ_a is a limit point of Z from the right.
4. Deduce that the zero set of Brownian motion is a.s. a perfect set: it is closed with no isolated point.
5. Write $Z(B)^c = \bigcup_n (l_n, r_n)$ a countable union of disjoint intervals. Show that almost surely, (r_n) is dense in $Z(B)$.
6. Let $R(\omega) = \{t : \omega(t) > 0\}$. Show that with probability one, $Z(\omega) \subset \overline{R(\omega)}$.

Exercise 2.9 Hitting time of 0 starting from $x \neq 0$
Using the notations of Exercise 2.1,

1. For $x \neq 0$, show that

$$P^x(\tau_1 \leq t) = \int_0^t \frac{|x|}{\sqrt{2\pi z^3}} e^{-\frac{x^2}{2z}} \, dz , \quad t \geq 0$$

2. Show that the density of τ_2 under P^0 is:

$$t \mapsto \frac{1}{\pi t \sqrt{t-1}} , \quad t > 1$$

Exercise 2.10 Maximum of Brownian motion with negative drift I
For $a > 0$, let $\tau_a = \inf\{t > 0 : B(t) = t + a\}$.

1. Use the strong Markov property to show that for $a, b > 0$:

$$\mathbb{P}(\tau_{a+b} < \infty | \tau_a < \infty) = \mathbb{P}(\tau_b < \infty).$$

2. Show that the random variable $\sup_{t \geq 0}\{B(t) - t\}$ has an exponential distribution.

Dirichlet Problem, continuous time martingales and stochastic integral

Exercise 3.1 Radial Dirichlet problem on the annulus
Let $d \geq 2$ be an integer.

1. Find all functions h which are harmonic on $\mathbb{R}^d \setminus \{0\}$ and radial (i.e $h(x) = g(|x|)$ for some $g : \mathbb{R}_+^* \to \mathbb{R}$).
2. Let $D = \{x \in \mathbb{R}^d \ : \ r_1 < |x| < r_2\}$ and f defined on ∂D by $f = 1$ on $\{|x| = r_2\}$ and $f = 0$ on $\{|x| = r_1\}$. Show that the solution h_1 to the Dirichlet problem on D with boundary condition f is given by:
$$h_1(x) = \begin{cases} \frac{\ln|x| - \ln r_1}{\ln r_2 - \ln r_1} & \text{if } d = 2 \\ \frac{\frac{1}{|x|^{d-2}} - \frac{1}{r_1^{d-2}}}{\frac{1}{r_2^{d-2}} - \frac{1}{r_1^{d-2}}} & \text{if } d \geq 3 \end{cases}$$
3. Show that for $x \neq 0$, almost surely Brownian motion in $\mathbb{R}^d$ started from 0 never hits x.
4. Show that Brownian motion is neighborhood recurrent in dimension $d = 2$: for any open set O, almost surely B hits O.
5. Show that Brownian motion (started from 0) is neighborhood transient in dimension $d \geq 3$ for any open set O not containing 0, with positive probability, Brownian motion never hits O.

Exercise 3.2 Compound Poisson process
Let N_t be a Poisson process of intensity λ on $\mathbb{R}_+$ and $(Y_j)_{j \geq 1}$ a sequence of i.i.d integrable random variables independent from N. Let $X_t = \sum_{i=1}^{N_t} Y_j$. Show that

- $X_t - \lambda t \mathbb{E}[Y_1]$ is a martingale,
- X is a Lévy process: it is càdlàg, has independent and stationary increments.

Exercise 3.3 Maximum of Brownian motion with negative drift II
For $a, b > 0$, let $\tau = \inf\{t > 0 : \ B(t) = a + bt\}$.

1. Let $M(t) = \exp(\alpha B(t) - \alpha^2 \frac{t}{2})$. Show that M is a martingale.
2. Show that for $\lambda > 0$:
$$\mathbb{E}[e^{-\lambda \tau} \mathbf{1}_{\tau < \infty}] = \exp(-a(b + \sqrt{b^2 + 2\lambda}))$$
3. Compute the distribution of $\sup_{t \geq 0} \{B_t - bt\}$.

Exercise 3.4
Compute the mean and variance of $\int_0^t B_s^2 \, dB_s$.

Exercise 3.5 Covariation process
For continuous square integrable semimartingales X and Y, define the covariation of X and Y as

$$\langle X, Y \rangle_t = \frac{1}{2} \langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t$$

1. Show that if X is a continuous martingale with zero quadratic variation, then X is constant.
2. Show that $\langle X, Y \rangle$ is the only bounded variation process A such that $XY - A$ is a martingale.
3. A process X is a square integrable continuous semimartingale if it can be written as

$$X_t = X_0 + M_t + A_t$$

   where M is a square integrable continuous martingale and A is a bounded variation process. If $Y_t = Y_0 + N_t + B_t$ is another square integrable continuous semimartingale, with N its martingale part, one defines

$$\langle X, Y \rangle_t := \langle M, N \rangle_t \, .$$

   Show that if Y is a bounded variation process $\langle X, Y \rangle = 0$ for every square integrable continuous semimartingale X.

4. Let M and N be square integrable martingales, H and G progressively measurable processes such that

$$\mathbb{E}[\int_0^T G_s^2 \, d\langle M \rangle_s] < \infty \text{ and } \mathbb{E}[\int_0^T H_s^2 \, d\langle N \rangle_s] < \infty$$

   for every T. If $X_t = \int_0^t G_s \, dM_s$ and $Y_t = \int_0^t H_s \, dN_s$, compute the covariation of X and Y.

Exercise 3.6 Stochastic exponential
Let $Y_t^1, \ldots, Y_t^d$ be continuous semimartingales which admit quadratic variations. The covariation is defined in Exercise 3.5. Here is Itô's formula for a C^2 function F from $\mathbb{R}^d$ to $\mathbb{R}$ of $Y_t = (Y_t^1, \ldots, Y_t^d)$:

$$\begin{aligned} F(Y_t^1, \ldots, Y_t^d) - F(Y_0^1, \ldots, Y_0^d) &= \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(Y_s^1, \ldots, Y_s^d) \, dY_s^i + \\ & \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(Y_s^1, \ldots, Y_s^d) \, d\langle Y^i, Y^j \rangle_s \end{aligned}$$

provided all the integrals on the right are well-defined. Let M_t be a continuous square integrable martingale and $\lambda \in \mathbb{R}$. Let $X_t = \exp\left(i\lambda M_t + \frac{\lambda^2}{2} \langle M \rangle_t\right)$. Show that

- X_t is a martingale,
- $X_t = X_0 + i\lambda \int_0^t X_s \, dM_s$.

Exercise 3.7 Lévy's characterization of Brownian motion

1. Let $X = (X^1, \dots, X^d)$ be a continuous square integrable martingale such that $X_0 = 0$ and

$$\langle X^i, X^j \rangle_t = t\mathbf{1}_{i=j}$$

Using Exercise 3.6, show that for $\xi \in \mathbb{R}^d$,

$$\mathbb{E}(e^{i\xi\cdot X_t + \frac{1}{2}|\xi|^2 t}|\mathcal{F}_s) = e^{i\xi\cdot X_s + \frac{1}{2}|\xi|^2 s}$$

2. Show that X is a d-dimensional Brownian motion.

Exercise 3.8 Conformal invariance of Brownian motion

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function whose derivative does not vanish and suppose for the sake of normalization that $f(0) = 0$ and write $f = f_1 + if_2$ its decomposition in real and imaginary parts. Let $B(t) = B_1(t) + iB_2(t)$ with B_1 and B_2 independent standard Brownian motions. Let $X^i_t = f_i(B(t))$ for $i = 1, 2$ and $X = (X^1, X^2)$. We shall suppose[1] that $\mathbb{E}[\int_0^t |f'(B(s))|^2 ds]$ is finite for any t.

1. Show that (seen as a process in $\mathbb{R}^2$ or in $\mathbb{C}$) X is a continuous martingale and:

$$\langle X^i, X^j \rangle_t = \int_0^t |f'(B(s))|^2 ds \mathbf{1}_{i=j}$$

2. Show that $\int_0^t |f'(B(s))|^2 ds$ goes to infinity almost surely as t goes to infinity.

3. Let $\sigma(t) = \inf\{u \geq 0 \ : \ \int_0^u |f'(B(s))|^2 ds \geq t\}$. Show that $X_{\sigma(t)}$ is a Brownian motion.

[1] This assumption can be dropped using the notion of local martingale

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Exercise 4.1 Percolation sur un graphe à degrés bornés
On considère un graphe $G = (V, E)$ où V est dénombrable et les degrés des sommets sont inférieurs à $\Delta \geq 1$. Pour $p \in [0, 1]$, on note $\mathbb{P}_p$ la mesure produit $\mathcal{B}(p)^{\otimes E}$ sur $\{0, 1\}^E$. Pour $\omega \in \{0, 1\}^E$ et x un élément de V on note $C_x(\omega)$ l'ensemble des sommets de la composante connexe de x dans le graphe $G(\omega) := (V, \{e \in E \ : \ \omega(e) = 1\})$. On note $\mathrm{diam}(A)$ le diamètre d'une partie $A \subset V$ pour la distance de graphe sur G.

1. En utilisant une borne d'union, montrer que :

$$\forall n \in \mathbb{N}, \quad \mathbb{P}_p(\mathrm{diam}(C_x(\omega)) \geq n) \leq \sum_{m \geq n/2} p^m \Delta(\Delta - 1)^{m-1} .$$

2. En déduire que si $p < \frac{1}{\Delta - 1}$, pour $\mathbb{P}_p$-presque tout ω, toutes les composantes connexes de $G(\omega)$ sont finies.

3. En utilisant une comparaison avec un processus de Galton-Watson, montrer que si $p < \frac{1}{\Delta - 1}$, il existe $C(p) > 0$ tel que:
$$\mathbb{P}(|C_x(\omega)| \geq n) \leq e^{-C(p)n}$$

Exercise 4.2 Construction graphique du processus d'exclusion
On veut construire le processus d'exclusion associé à la Q-matrice q sur un graphe $G = (V, E)$ à degrés bornés où V est dénombrable. On suppose que $q(x, y) = 0$ si x et y ne sont pas voisins.

1. Montrer que si $\sup_{x,y}\{q(x, y) + q(y, x)\} < +\infty$, la construction graphique détaillée dans le cours permet de construire simultanément pour toutes les configurations initiales un processus de Markov sur $\{0, 1\}^V$ (le processus d'exclusion).

2. Montrer que pour toute fonction f locale (i.e ne dépendant que d'un ensemble fini de coordonnées),

$$Lf(\eta) := \frac{d}{dt}\mathbb{E}^{\eta}[f(\eta_t)]\bigg|_{t=0} = \sum_{x,y} q(x, y)\eta(x)(1 - \eta(y))(f(\eta^{x\to y}) - f(\eta)) .$$

Exercise 4.3 Noisy voter model
On considère le processus de Markov à temps continu défini informellement comme suit. L'espace d'états est $\{0,1\}^{\mathbb{Z}^d}$ et on note η une configuration. Pour chaque couple de voisins (x,y), le site x transmet son état au site y à taux 1. De plus, si $\eta(x)=0$ (resp. si $\eta(x)=1$), $\eta(x)$ change d'état à taux β (resp. à taux δ).

1. En utilisant un argument de percolation, montrer qu'un tel processus existe.
2. Calculer, pour toute fonction f locale (i.e ne dépendant que d'un ensemble fini de coordonnées), $\frac{d}{dt}\mathbb{E}^\eta[f(\eta_t)]\big|_{t=0}$.

Exercise 4.4 Couplage pour le modèle d'exclusion simple symétrique fini
On considère le modèle d'exclusion simple symétrique sur $\mathbb{Z}$. On se donne deux configurations initiales A et B finies (i.e avec un nombre fini de particules) telles que $|A|=|B|=n$ et $|A\cap B|=n-1$.

1. Utiliser un couplage avec des particules de première et de seconde classe pour montrer que presque sûrement, $A_t=B_t$ pour t assez grand.
2. Utiliser ce résultat pour montrer que les mesures invariantes de SSEP sont échangeables.

Exercise 4.5 Des mesures invariantes extrêmales pour ASEP sur $\mathbb{Z}$
On considère le processus d'exclusion simple asymétrique (η_t)sur $\mathbb{Z}$, où la marche sous-jacente est constituée de sauts à gauche (resp. à droite) avec probabilité q (resp. $1-q=p$), avec $0<q<p$. Soit

$$\Lambda=\{\eta\in\{0,1\}^{\mathbb{Z}}\ :\ \sum_{x<0}\eta(x)<\infty \text{ et } \sum_{x>0}1-\eta(x)<\infty\}$$

et pour tout $n\in\mathbb{Z}$,

$$\Lambda_n=\{\eta\in\Lambda\ :\ \sum_{x<n}\eta(x)=\sum_{x\geq n}1-\eta(x)\}\ .$$

1. Montrer que Λ est dénombrable et égal à $\bigcup_{n\in\mathbb{Z}}\Lambda_n$.
2. Montrer que sur Λ, η_t est une chaîne de Markov dont les classes irréductibles sont les Λ_n.
3. On note, pour $x\in\mathbb{Z}$, $\alpha(x)=\frac{\left(\frac{p}{q}\right)^x}{1+\left(\frac{p}{q}\right)^x}$ et $\nu=\bigotimes_{x\in\mathbb{Z}}\mathcal{B}(\alpha(x))$. En utilisant ν, montrer que, restreinte à Λ_n, η_t admet une mesure de probabilité invariante.
4. En déduire une famille de mesures invariantes extrêmales $(\pi_n)_{n\in\mathbb{Z}}$ deux à deux distinctes et distinctes des mesures produits.

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Exercise 5.1 Attractivité
Soit $\mathcal{S}$ dénombrable. On munit $\Omega = \{0,1\}^{\mathcal{S}}$ de l'ordre partiel usuel, coordonnée par coordonnée. On note $\mathcal{M}$ l'ensemble des fonctions croissantes et continues (pour la topologie produit) sur Ω. On dit que $\mu_1 \leq \mu_2$, pour deux mesures de probabilité sur Ω si $\int f\, d\mu_1 \leq \int f\, d\mu_2$ pour toute $f \in \mathcal{M}$. On dit enfin qu'un semigroupe de probabilités P_t sur $C(S)$ est attractif si

$$\mu_1 \leq \mu_2 \Rightarrow \forall t \geq 0 \; \mu_1 P_t \leq \mu_2 P_t \; .$$

1. Montrer que P_t est attractif si et seulement si:

$$f \in \mathcal{M} \Rightarrow P_t f \in \mathcal{M}$$

2. Montrer que le noisy voter model et le processus d'exclusion sont attractifs.
3. On considère P_t un semi-groupe attractif. On note δ_0 (resp. δ_1) la mesure de Dirac concentrée sur la configuration constante égale à 0 (resp. égale à 1). Montrer que:
   (a) $\delta_0 P_s \leq \delta_0 P_t$ et $\delta_1 P_s \geq \delta_1 P_t$ pour $s \leq t$.
   (b) $\underline{\nu} = \lim_{t\to\infty} \delta_0 P_t$ et $\overline{\nu} = \lim_{t\to\infty} \delta_1 P_t$ existent et sont stationnaires.
   (c) Pour toute mesure μ sur Ω, $\delta_0 P_t \leq \mu \leq \delta_1 P_t$.
   (d) Toute limite ν de μP_t le long d'une suite t_n tendant vers l'infini satisfait $\underline{\nu} \leq \nu \leq \overline{\nu}$.
   (e) P_t est ergodique si et seulement si $\underline{\nu} = \overline{\nu}$.

Exercise 5.2 Noisy voter model II
On s'intéresse au noisy voter model avec $\delta+\beta > 0$, et on note P_t son semigroupe. Soit ν une mesure de probabilité sur $\Omega = \{0,1\}^{\mathbb{Z}^d}$ telle que $\nu(\eta \; : \; \eta(x) = 1) =: c$ ne dépende pas de x. On note $u(t,x) = \nu P_t(\eta \; : \; \eta(x) = 1)$.

1. Montrer que:
$$\partial_t u(t,x) = -u(t,x)(1+\delta+\beta) + \beta + \sum_y p(x,y) u(t,y)$$
puis que $u(t,x) = \frac{\beta}{\delta+\beta} + \left(c - \frac{\beta}{\delta+\beta}\right) e^{-(\delta+\beta)t}$.
2. En déduire que $\underline{\nu}(\eta \; : \; \eta(x) = 1) = \overline{\nu}(\eta \; : \; \eta(x) = 1) = \frac{\beta}{\delta+\beta}$.
3. Montrer que le noisy voter model avec $\delta+\beta > 0$ est ergodique. Que se passe-t-il si $\delta+\beta = 0$?

Exercise 5.3 Limite hydrodynamique pour des particules sans interaction
On considère des particules sans interaction sur le tore discret $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$ faisant des marches aléatoires de moyenne nulle et de variance finie. Plus précisément, on suppose que chaque particule suit une marche aléatoire a temps continu: à taux $p(x,y)$, elle saute de $x \in \mathbb{T}_N^d$ vers $y \in \mathbb{Z}^d$ modulo N, p vérifiant $p(x,y) = p(0, y-x)$, $\sum_y p(0,y) = 1$, $\sum_{y\in\mathbb{Z}^d} yp(0,y) = 0$. On note $\sigma_{i,j} = \sum_{y\in\mathbb{Z}^d} y_i y_j p(0,y)$. On note les configurations dans $\mathbb{N}^{\mathbb{T}_N^d}$. Pour un profil ρ du tore $\mathbb{T}^d = [0,1[^d = (\mathbb{R}/\mathbb{Z})^d$ dans $\mathbb{R}^+$, on note ν_ρ^N la mesure produit $\bigotimes_{x\in\mathbb{T}_N^d} \mathcal{P}(\rho(x/N))$ sur $\mathbb{N}^{\mathbb{T}_N^d}$.

1. Déterminer la loi de η_t lorsque η_0 a pour loi ν_{ρ_0}.
2. Montrer que pour tout $\alpha > 0$, la mesure de Poisson ν_α de paramètre α sur $\mathbb{N}^{\mathbb{T}_N^d}$ est invariante.
3. Pour $u \in \mathbb{T}^d$, déterminer la limite en loi de $\eta_{tN^2}(\lfloor Nu \rfloor)$ lorsque N tend vers l'infini.
4. On suppose que ρ_0 est continue sur $\mathbb{T}^d$. Montrer que le profil limite $\rho(t,u)$ vérifie:

$$\begin{cases} \partial_t \rho & = \sum_{1\leq i,j\leq d} \sigma_{i,j} \partial^2_{u_i,u_j} \rho \\ \rho(0,.) & = \rho_0 \end{cases}$$

5. En notant $\pi_t^N = \frac{1}{N^d} \sum_i \delta_{X_t^i/N}$ où X_t^i est la position de la i-ème particule au temps t, montrer que $\pi_{tN^2}^N$ converge, lorsque N tend vers l'infini, vers la mesure de densité $\rho(t,.)$.

Exercise 5.4 Zero-range et exclusion
Le processus "zero-range" a pour espace d'états $\mathbb{N}^\mathbb{Z}$: en chaque site, il y a un nombre fini de particules. En chaque site x à taux 1, une particule est choisie au hasard parmi les $\eta(x)$ particules du site et saute à droite avec probabilité p et à gauche avec probabilité $q = 1-p$. On note $\mathcal{S} = \{0,1\}^\mathbb{Z}$ l'espace d'états du processus d'exclusion et $\mathcal{S}_\infty$ le sous-ensemble de configurations de $\mathcal{S}$ qui ont une infinité de particules à gauche et à droite de zéro et qui ont une particule en 0 (une "particule marquée").

1. Définir une bijection entre $\mathbb{N}^\mathbb{Z}$ et $\mathcal{S}_\infty$ où les particules du premier correspondent aux trous du deuxième.
2. Vérifier que l'image par cette bijection du processus d'exclusion simple vu depuis une particule marquée à partir d'une configuration de $\mathcal{S}_\infty$ est bien le processus zero-range.
3. En déduire que les mesures produits de lois géométriques de même paramètre sont des mesures invariantes pour le processus zero-range.